MATH 1224: Exam 3 Exercises

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August 1, 2023

The following exercises are designed to aid your review in preparing for this exam. The inclusion of an exercise in this set may not accurately represent its presence or difficulty of an actual exam problem.

You Should Know...

The following items are key results you should know.

p-series. All the sequences $\frac{1}{n^p}$ converge to zero. The corresponding series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if p > 1, and diverges otherwise. For some specific examples, the harmonic series (for the case $p = 1$) is divergent, but the case for $p = 2$ is convergent.[1]

Theorem 1 (Divergence Test)**.** *Suppose* $a_n \not\to 0$. *Then the sum* $\sum a_n$ *diverges.*

Note that this not give a sufficient condition for convergence. In fact, $a_n \to 0$ is a necessary condition for convergence (if a sum converges, its terms go to zero).

Theorem 2 (Direct Comparison Test for Series)**.** *Suppose* a_n *and* b_n *are sequences of positive terms satisfying* $a_n \leq b_n$ *for each* n.

- *If* $\sum b_n$ *is convergent, then so is* $\sum a_n$.
- *If* $\sum a_n$ *is divergent, then so is* $\sum b_n$.

This is the discrete analogue of the test for improper integrals.

[1]The value of that sum is $\frac{\pi^2}{6}$ by the way.

If no direct comparison can be made, the below theorem can be used.

Theorem 3 (Limit Comparison Test)**.** *Suppose a_n and b_n are two positive sequences with $b_n \neq 0$ for any n. If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

where L is positive and finite, then the sums $\sum a_n$ and $\sum b_n$ behave the same, i.e., they both converge or diverge. If $L = 0$ and $\sum_{b_n}$ converges, then $\sum_{a_n}$ converges. If $L = \infty$ and $\sum_{b_n}$ diverges, then $\sum_{a_n}$ diverges.

Note that in the Limit Comparison Test, we are only concerned with asymptotic growth. When using this, you should discard any lower order terms - that is, terms whose contribution don't affect the limit.[2] To see this, we note that, for example,

$$\frac{1}{n^4 - 3n} \sim \frac{1}{n^4}$$

as the contribution of $3n$ does not greatly affect the limit. (We use the notation $f \sim g$, read as "f is asymptotic to g" to say that

$$\frac{f(n)}{g(n)} \to L \text{ as } n \to \infty$$

for some constant L. Students in computer science will be familiar with this during a course in the analysis of algorithms - CS 3343 and CS 5633.) This test is also very powerful for comparing against known p-series and geometric series.

The Integral Test is easy to use, but requires a lot of hypotheses to be satisfied.

Theorem 4 (Integral Test)**.** *Suppose a_n is a sequence of positive and decreasing terms. Set $a_n = f(n)$, where f is decreasing. Then*

$$\sum_{n=n_0}^{\infty} a_n \text{ converges } \iff \int_{n_0}^{\infty} f(x)\,dx \text{ converges.}$$

In order to use the integral test, you should show via inequalities or derivatives that the sequence is decreasing. That is, you should show one of the following:

- For each $u < v$ we have $f(u) > f(v)$.
- $f'(x) < 0$ for each $x > N$.

As a final note, we remark that the Integral Test does not give the value of the infinite sum, but instead an upper bound on its value. Often we aren't concerned about the exact value of the sum, but we can arrive at estimates using techniques beyond the scope of this course.[3]

[2] This is an informal phrasing.

[3] For example, the Euler-MacLaurin summation formula.

The ratio test is best used when factorials and exponents are present. It is perhaps the most powerful test.

Theorem 5 (Ratio Test)**.** *Suppose*

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

- *If $L < 1$, the sum $\sum a_n$ is absolutely convergent.*
- *If $L > 1$, the sum $\sum a_n$ is divergent.*
- *The test is inconclusive if $L = 1$.*

Theorem 6 (Root Test)**.** *Suppose*

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(|a_n|\right)^{\frac{1}{n}}.$$

- *If $L < 1$, the sum $\sum a_n$ is absolutely convergent.*
- *If $L > 1$, the sum $\sum a_n$ is divergent.*
- *The test is inconclusive if $L = 1$.*

The root test is most powerful when the index appears in an exponent. It can also be shown that the ratio and root tests will produce the same conclusions.

Theorem 7 (Alternating Series Test)**.** *Consider the sum*

$$S = \sum (-1)^n b_n.$$

If b_n is positive, decreasing, and has limit zero, then S is convergent.

This should be the starting point for determining if a sum converges absolutely or conditionally.

1 Sequences

Determine if each sequence is convergent or divergent. If the sequence is convergent, find the value of the limit.

1. $a_n = \frac{1-2n}{1+2n}$
2. $b_n = \frac{n+(-1)^n}{n}$
3. $c_n = \frac{\sin n}{n^2}$ (Hint: you may want to use the Squeeze Theorem.)
4. $d_n = \left(\frac{2}{3}\right)^n$
5. $x_n = \sqrt{\frac{2n}{n+1}}$

2 Infinite Series

For each series, find a formula for the partial sum, and use that to find the limit. (All sums in this section converge.)

1. $\sum_{n=1}^{\infty} \frac{4}{n(n+1)}$

2. $\sum_{n=1}^{\infty} \left(-\frac{2}{5}\right)^n$

3. $\sum_{n=2}^{\infty} \frac{6}{n^2-1}$ (Hint: Partial fractions.)

4. $\sum_{n=1}^{\infty} \frac{2^{n+1}}{3^{n-1}}$ (Hint: Peel away.)

3 Convergence Tests

Determine if each sum is convergent or divergent. State clearly the test you use. If the sum is alternating, determine if the sum is absolutely convergent.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

2. $\sum_{n=2}^{\infty} \frac{\ln n^2}{n}$ (Hint: A log property can make this look a bit nicer.)

3. $\sum_{n=1}^{\infty} \frac{2^n}{3^n-4}$

4. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$

5. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$ (Hint: There are three ways to work this that seem natural.)

6. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\ln n}$

7. $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(4n)^n}{(2n)!}$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n+2}{n^3}\right)$

11. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{\sin n}{n\sqrt{n}}\right)$ (Hint: Use the fundamental inequality $|\sin x| \leq 1$ for each x.)

12. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$