Exam 3 Guide Solutions.

1. Sequences.

1.1. $\lim_{n\to\infty} \frac{1-2n}{1+2n} = -1$. Thus a_n is convergent.

1.2. $\lim_{n\to\infty} \frac{n+(-1)^n}{n} = \lim_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1$. Thus b_n is convergent.

1.3. By the squeeze theorem,

$$\lim_{n\to\infty} \left|\frac{\sin n}{n^2}\right| \leq \lim_{n\to\infty} \frac{1}{n^2} = 0.$$ Thus c_n is convergent.

(Remark: Note that by comparison $\sum_{n\geq 1} |c_n| < \infty$, i.e., the sum is absolutely convergent - and hence convergent.)

1.4. $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$, so d_n is convergent.

1.5. As $\sqrt{n}$ is continuous, we may pass the limit through. So,

$$\lim_{n\to\infty} \sqrt{\frac{2n}{n+1}} = \sqrt{\lim_{n\to\infty} \frac{2n}{n+1}} = \sqrt{2}.$$

Thus x_n is convergent.

2. Infinite Series.

Recall for geometric series, the k^{th} partial sum is

$$\sum_{n=1}^{k} ar^{n-1} = \frac{a(1-r^k)}{1-r}$$

(note that exponent in the summand is $n-1$ and the sum starts at 1). The limit as $k \to \infty$ exists iff $|r| < 1$. That is,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

Don't forget the exponent rules

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad a^{n+1} = a^n \cdot a.$$

Finally, some telescoping sums require partial fractions: let K be a real number. There exist A, B such that

$$\frac{K}{(n+a)(n+b)} = \frac{A}{n+a} + \frac{B}{n+b}.$$

(See partial fractions notes.)

2.1. First decompose into partial fractions.

$$\frac{4}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

This yields $4 = A(n+1) + Bn$. When $n=0$, $A=4$ and when $n=-1$, $B=-4$.

Consider the partial sum S_k:

$$S_k = \sum_{n=1}^{k} \frac{4}{n(n+1)} = \sum_{n=1}^{k}\left(\frac{4}{n} - \frac{4}{n+1}\right)$$

$$= \left(\frac{4}{1} - \frac{4}{2}\right) + \left(\frac{4}{2} - \frac{4}{3}\right) + \cdots + \left(\frac{4}{k-1} - \frac{4}{k}\right) + \left(\frac{4}{k} - \frac{4}{k+1}\right)$$

$$= 4 - \frac{4}{k+1}.$$

Observe that $S_k \to 4$. The infinite sum is 4.

2.2. Observe that $\left(-\frac{2}{5}\right)^n = \left(-\frac{2}{5}\right)\left(-\frac{2}{5}\right)^{n-1}$. Identify $a = -\frac{2}{5}$ and $r = -\frac{2}{5}$.
Now the k^{th} partial sum is

$$S_k = \sum_{n=1}^{k}\left(-\frac{2}{5}\right)\left(-\frac{2}{5}\right)^{n-1} = \frac{-\frac{2}{5}\left(1 - \left(-\frac{2}{5}\right)^k\right)}{\left(1 - \left(-2/5\right)\right)} = -\frac{2}{7}\left(1 - \left(-\frac{2}{5}\right)^k\right).$$

Now $S_\infty = \lim_{k \to \infty} S_k = -\frac{2}{7}.$

2.3. Write, with partial fractions,

$$\frac{6}{n^2-1} = \frac{3}{n-1} - \frac{3}{n+1}$$

and so

$$S_k = \sum_{n=2}^{k} \frac{6}{n^2-1} = \sum_{n=2}^{k}\left(\frac{3}{n-1} - \frac{3}{n+1}\right)$$

$$= \left(\frac{3}{1} - \cancel{\frac{3}{3}}\right) + \left(\frac{3}{2} - \cancel{\frac{3}{4}}\right) + \left(\cancel{\frac{3}{3}} - \cancel{\frac{3}{5}}\right) + \left(\cancel{\frac{3}{4}} - \cancel{\frac{3}{6}}\right)$$

$$+ \cdots + \left(\cancel{\frac{3}{k-3}} - \cancel{\frac{3}{k-1}}\right) + \left(\cancel{\frac{3}{k-2}} - \frac{3}{k}\right) + \left(\cancel{\frac{3}{k-1}} - \frac{3}{k+1}\right)$$

$$= 3 + \frac{3}{2} - \frac{3}{k} - \frac{3}{k+1}.$$

Now $S_k \xrightarrow{k\to\infty} \frac{9}{2}$.

2.4. First,

$$\frac{2^{n+1}}{3^{n-1}} = \frac{2^2 \cdot 2^{n-1}}{3^{n-1}} = 4\left(\frac{2}{3}\right)^{n-1}.$$

So $a = 4$, $r = 2/3$. Now $S_k = \dfrac{4\left(1-\left(\frac{2}{3}\right)^k\right)}{(1-2/3)} = 12\left(1-\left(\frac{2}{3}\right)^k\right)$, and

$S_k \xrightarrow{k\to\infty} 12$.

3. Convergence tests

When possible, multiple solutions will be given.

3.1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

Solution 1. Let $f(x) = \frac{1}{x^2+1}$. Notice that $f'(x) = \frac{-2x}{(x^2+1)^2} < 0$ for each $x \geqslant 1$. So f is decreasing on $[1, \infty)$. Certainly f is everywhere positive. By the Integral Test,

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \tan^{-1} x \Big|_1^{\infty}$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4},$$

Thus, $\sum_{n\geqslant 1} \frac{1}{n^2+1}$ is convergent.

Solution 2. Use limit comparison against $\frac{1}{n^2}$. Indeed,

$$\frac{\frac{1}{n^2+1}}{\frac{1}{n^2}} = \frac{n^2}{n^2+1} \xrightarrow{n\to\infty} 1.$$

As $\sum_{n\geqslant 1} \frac{1}{n^2}$ converges, we conclude that $\sum_{n\geqslant 1} \frac{1}{n^2+1}$ converges.

<u>Solution 3.</u> Notice for each n,

$$n^2 < n^2+1 \implies \frac{1}{n^2+1} < \frac{1}{n^2}.$$

By comparison, as $\sum_{n\geq 1} \frac{1}{n^2}$ converges, $\sum_{n\geq 1} \frac{1}{n^2+1}$ also converges.

<u>3.2.</u> $\sum_{n\geq 2} \frac{\ln n^2}{n}$

<u>Solution 1.</u> Notice that $\ln n^2 = 2\ln n$. Now, let $f(x) = \frac{\ln x}{x}$. Then

$$f'(x) = \frac{1-\ln x}{x^2} < 0 \text{ for } x > e.$$

Thus,

$$\int_3^\infty \frac{2\ln x}{x}\,dx = (\ln x)^2\Big|_3^\infty = \infty.$$

Thus $\sum_{n\geq 2} \frac{\ln n^2}{n}$ diverges by the integral test.

<u>Solution 2:</u> By comparing against $\frac{1}{n}$, and as $\sum_{n\geq 2}\frac{1}{n}$ diverges,

$\lim_{n\to\infty} \frac{2\ln n}{n}\cdot\frac{n}{1} = \lim_{n\to\infty} 2\ln n = \infty.$ Thus $\sum_{n\geq 2}\frac{\ln n^2}{n}$ diverges.

2.3. $\displaystyle\sum_{n\geq 1} \frac{2^n}{3^n-4}$.

Solution 1. The sum $\displaystyle\sum_{n\geq 1}\left(\frac{2}{3}\right)^n$ converges, as it is a geometric series with $|r|<1$. Now, by limit comparison,

$$\lim_{n\to\infty} \frac{2^n}{3^n-4}\cdot\frac{3^n}{2^n} = \lim_{n\to\infty}\frac{3^n}{3^n-4} = 1.$$

Thus the given sum converges.

2.4. $\displaystyle\sum_{n\geq 1}\frac{\sqrt{n}}{n^2+1}$.

Solution 1. By comparison, as $n^2 < n^2+1$ implies $\frac{1}{n^2+1} < \frac{1}{n^2}$,

$$\sum_{n\geq 1}\frac{\sqrt{n}}{n^2+1} \leq \sum_{n\geq 1}\frac{\sqrt{n}}{n^2}$$

$$= \sum_{n\geq 1}\frac{1}{n^{3/2}}.$$

The last sum is a p-series with $p = 3/2$, which is convergent. Thus the original sum converges.

Solution 2. Use limit comparison against $\frac{1}{n^{3/2}}$.

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n^2+1}\cdot n^{3/2} = \lim_{n\to\infty}\frac{n^2}{n^2+1} = 1.$$

2.5. $\sum_{n\geq 1} \left(\frac{n}{3n+1}\right)^n$.

Solution 1. The root test is natural to use.

$$\lim_{n\to\infty}\left[\left(\frac{n}{3n+1}\right)^n\right]^{1/n} = \lim_{n\to\infty}\frac{n}{3n+1} = \frac{1}{3}.$$

The sum converges.

Solution 2. The ratio test is also natural.

$$\lim_{n\to\infty}\left(\frac{(n+1)}{3(n+1)+1}\right)^{n+1}\cdot\left(\frac{3n+1}{n}\right)^n = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\cdot\frac{(3n+1)^n}{(3n+4)^{n+1}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^n}{n^n}\cdot\frac{(n+1)}{(3n+4)}\cdot\frac{(3n+1)^n}{(3n+4)^n}$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{n+1}{3n+4}\right)\cdot\left(\frac{3n+1}{3n+4}\right)^n$$

$$= \frac{1}{3}.$$

The sum converges.

Solution 3. Use limit comparison against $\left(\frac{1}{3}\right)^n$, which is summable. The solution is omitted and left as an exercise.

3.6. $\sum_{n \geq 2} \frac{1}{\sqrt{n} \ln n}$.

First observe that $\sqrt{n} < n$ for $n \geq 2$, and then we have the inequalities

$$\frac{1}{n} < \frac{1}{\sqrt{n}} \Rightarrow \frac{1}{n \ln n} \leq \frac{1}{\sqrt{n} \ln n}.$$

Notice that $\frac{1}{n \ln n}$ satisfies the hypotheses of the integral test. Now,

$$\int_2^\infty \frac{1}{x \ln x}\, dx = \ln(\ln x)\Big|_2^\infty = \infty$$

So $\sum_{n \geq 2} \frac{1}{n \ln n}$ is divergent. By comparison, the original sum is divergent.

3.7. $\sum_{n \geq 1} \frac{10^n}{n!}$

Ripe for the ratio test.

$$\lim_{n \to \infty} \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} = \lim_{n \to \infty} \frac{10\, n!}{(n+1)\, n!}$$

$$= \lim_{n \to \infty} \frac{10}{n+1} = 0.$$

The sum converges.

3.8. $\sum_{n\geq 1} \frac{2^n}{n^2}$.

Solution 1. Use the ratio test.

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = \lim_{n\to\infty} 2\left(\frac{n}{n+1}\right)^2$$

$$= \lim_{n\to\infty} 2\left(1-\frac{1}{n+1}\right)^2$$

$$= 2.$$

The sum diverges.

Solution 2. Although slightly trickier, the root test is an option.

$$\lim_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{1/n} = \lim_{n\to\infty} \frac{2}{n^{2/n}} = L.$$

Now,

$$\ln L = \lim_{n\to\infty} \ln\left(\frac{2}{n^{2/n}}\right)$$

$$= \lim_{n\to\infty} \ln 2 - \frac{2}{n}\ln n$$

$$= \ln 2 \quad \left(\text{as } \frac{\ln n}{n} \to 0\right)$$

So $L=2$, and the sum diverges.

3.9. $\displaystyle\sum_{n\geq 1} \frac{(4n)^n}{(2n)!}$

<u>Solution 1.</u> Use the root test.

$$\lim_{n\to\infty} \left(\frac{(4n)^n}{(2n)!}\right)^{1/n} = \lim_{n\to\infty} \frac{4n}{\left[(2n)!\right]^{1/n}}$$

The remainder of the solution is omitted, as handling $\frac{1}{\sqrt[n]{(2n)!}}$ requires Stirling approximations.

<u>Solution 2.</u> The ratio test is slightly more tame.

$$\lim_{n\to\infty} \frac{(4(n+1))^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{(4n)^n} = \lim_{n\to\infty} \frac{(4n+4)^{n+1}}{(4n)^n} \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} \left(\frac{4n+4}{4n}\right)^n \cdot \frac{(4n+4)}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \left(\frac{4(n+1)}{4n}\right)^n \frac{4n+4}{4n^2+6n+2}$$

$$= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n \left(\frac{4n+4}{4n^2+6n+2}\right)$$

$$= 0$$

(notice $\left(1-\frac{1}{n}\right)^n \to e^{-1}$ and $\frac{4n+4}{4n^2+6n+2} \to 0$.) The sum converges.

3.10. $\sum_{n\geq 1} (-1)^{n+1}\left(\frac{n+2}{n^3}\right)$

Solution 1. This is an alternating sum. The sequence $\frac{n+2}{n^3}$ is positive and decreasing - to see this, observe that

$$\frac{n+2}{n^3} = \frac{1}{n^2} + \frac{2}{n^3} < \frac{1}{m^2} + \frac{2}{m^3}$$

for $m > n$ (use the reciprocal comparison). The sum is convergent by the alternating series test. Absolute convergence follows by the triangle inequality:

$$\sum_{n\geq 1}\left|(-1)^{n+1}\left(\frac{n+2}{n^3}\right)\right| \leq \sum_{n\geq 1} \frac{1}{n^2} + \frac{2}{n^3}$$

$$= \sum_{n\geq 1}\frac{1}{n^2} + \sum_{n\geq 1}\frac{2}{n^3} < \infty.$$

(both sums are convergent p-series.) Limit comparison against $\frac{1}{n^2}$ is also an option:

$$\lim_{n\to\infty} \frac{n+2}{n^3}\cdot\frac{n^2}{1} = \lim_{n\to\infty}\frac{n^3+2n^2}{n^3} = 1.$$

There is also the option of direct comparison: as $\frac{n+2}{n^3} < \frac{2n}{n^3} = \frac{2}{n^2}$, and $\frac{2}{n^2}$ is summable, the original sum converges.

Remark: The ratio test fails directly.

3.11. $\sum_{n\geq 1} (-1)^n \frac{\sin n\pi}{n\sqrt{n}}$.

Using the estimate $|\sin x| \leq 1$,

$$\left|\sum_{n\geq 1} (-1)^n \frac{\sin n\pi}{n\sqrt{n}}\right| \leq \sum_{n\geq 1}\left|(-1)^n \frac{\sin n\pi}{n\sqrt{n}}\right|$$

$$\leq \sum_{n\geq 1} \frac{1}{n\sqrt{n}} < \infty$$

(by comparison against $\sum_{n\geq 1} \frac{1}{n^{3/2}}$.) The sum converges absolutely.

3.12. $\sum_{n\geq 1} (-1)^n \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$.

The root test demonstrates absolute convergence.

$$\lim_{n\to\infty} \left|(-1)^n \left(\frac{1}{n} - \frac{1}{n^2}\right)^n\right|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = 0.$$