The Euler-Maruyama Method for Stochastic Differential Equations

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One direct application of mathematics in the industrial setting is the need to solve various types of equations. Many of these equations are deterministic in nature, where the result will be the same each time. In fields where randomness is present, such as quantitative finance and physics, the task is now to understand the behavior of the solutions in the average case. In this paper, we describe a method to numerically solve stochastic differential equations and demonstrate the method with a relatively simple example.

To begin, we give a brief description of the mechanics of stochastic calculus. An extended treatment is given in an informal talk given to the Texas A&M University graduate math student organization [5], with formal treatments given in Kloeden and Platen [1], Øksendal [4], and Kuo [2]. Our fundamental object will be a Brownian motion, denoted by B_t. This is a stochastic process that satisfies the following properties:

1. $B_0 = 0$ almost surely.
2. When $0 \leq s < t$, the random variable $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$.
3. B_t has independent increments. Given a sequence of times $\{t_1, t_2, \dots, t_n\}$, the random variables $B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_{n-1}} - B_{t_n}$ are independent.
4. Almost every sample path of B_t is continuous.

One tool that will be needed in the stochastic calculus is integration. This is developed by modifying the Riemann-Stieltjes integral by allowing for the integrator to be a random object. Thus we are interested when it makes sense to write

$$\int_a^b f(t, X_t)\, dB_t$$

where X_t is some stochastic process. We can extend this further to when the integrator is not just a Brownian motion (e.g., a semimartingale), but such a treatment will not be considered here.

The stochastic calculus would not be complete without a variant of the chain rule from the ordinary calculus. The result below, attributed to Kiyosi Ito, is the main tool in solving stochastic differential equations.

Theorem 1 (Ito's Lemma)**.** *Let $f(t,x)$ be $\mathcal{C}^2([0,\infty)\times\mathbb{R})$. Then the process $X_t = f(t, B_t)$ has the following "derivative:"*

$$dX_t = \left(f_t + \frac{1}{2}f_{xx}\right)\,dt + f_x\,dB_t \tag{1}$$

Equivalently, we may write

$$X_t = X_0 + \int_0^t \left(f_s(s,B_s) + \frac{1}{2}f_{xx}(s,B_s)\right)\,ds + \int_0^t f_x(s,B_s)\,dB_s \tag{2}$$

Observe that the statement in equation (2) provides a method of expressing the solution to a stochastic differential equation as a sum of two integrals - the first is a deterministic integral and the second is a stochastic integral. This will be fundamental in exactly solving certain stochastic differential equations and also in developing our numerical scheme for solving these equations.

To motivate the construction of stochastic differential equations, we first consider an exponential growth model, written in the differential form.

$$dx = kx\,dt \tag{3}$$
$$x(0) = x_0$$

This equation admits the solution $x(t) = x_0e^{kt}$. This is readily seen by separating variables or using another method. However, if we introduce random noise to the system, say, by replacing k with $k + \alpha B_t$, then the equation to solve is

$$dX_t = kX_t\,dt + \alpha X_t\,dB_t \tag{4}$$

where the derivative was found by using equation (1). The solution to equation (4) is obtained by writing

$$\frac{dX_t}{X_t} = k\;dt + \alpha\;dB_t$$

and then integrating both sides on the interval $[0, t]$. Thus we obtain

$$\int_0^t \frac{dX_s}{X_s}\;ds = kt + \alpha B_t.$$

To evaluate the integral on the left, consider the computation of the differential $d(\ln X_t)$ by the Ito lemma. Eventually this yields the solution

$$X_t = X_0 \exp\left(\left(k - \frac{1}{2}\alpha^2\right) t + \alpha B_t\right).$$

While equation (4) could be easily solved by mimicking methods for ordinary differential equations, not all stochastic differential equations can be analytically solved. To determine the behavior of solutions to equations, such as the trajectory of the mean, the equation can be numerically solved over a large number of runs. The method used was devised by Gisiro Maruyama in his 1951 paper [3], and it builds on the usual forward Euler method for ordinary differential equations. The differential equation

$$\frac{dx}{dt} = f(t, x)$$

subject to the initial condition $x(0) = x_0$ can be numerically solved via the iterative scheme

$$x(t_n) = x_{n-1} + hf(t_n, x(t_n)) \tag{5}$$

where the step size is chosen as $h = t_n - t_{n-1}$. We assume this step size is uniform. We apply this to a stochastic differential equation

$$dX_t = u(X_t)\;dt + v(X_t)\;dB_t \tag{6}$$

in the following manner, adapted from Kloeden and Platen [1]: replace t by the increment $t+h$, and approximate the differentials by writing $dX_t \approx \frac{1}{h}(X_{t+h} - X_t)$ and $dB_t \approx \frac{1}{h}(B_{t+h} - B_t)$. Then the continuous time process in equation (6) can be approximated by the discrete-time Markov chain Y_t by the iteration

$$Y_{t+h} = Y_t + hu(Y_t) + v(Y_t)\left(B_{t+h} - B_t\right) \tag{7}$$

subject to the initial condition $Y_0 = X_0$. While there are stability and convergence considerations, they will be ignored in this paper.

To demonstrate the utility of the Euler-Maruyama method, consider the equation

$$dX_t = -0.8X_t\, dt + dB_t \tag{8}$$

subject to the initial condition equation $X_0 = 1$. Equation (8) can be solved analytically as follows: multiply by the integrating factor $e^{0.8t}$ to yield

$$e^{0.8t} dX_t = -0.8X_t e^{0.8t}\, dt + e^{0.8t}\, dB_t$$

which, after rearranging and using the product rule, gives

$$d\left(X_t e^{0.8t}\right) = e^{0.8t}\, dB_t.$$

Finally, after integrating both sides on $[0, t]$, the solution is

$$X_t = e^{-0.8t}\left(X_0 + \int_0^t e^{0.8s}\, dB_s\right). \tag{9}$$

The solution obtained in equation (9) will be the basis of comparison against the numerical estimated obtained by the iterations in equation (7), where $u(X_t) = -0.8X_t$ and $v(X_t) = 1$. The simulations are conducted via Python, where $h \in \{0.1, 0.02, 0.01, 0.005, 0.002, 0.001, 0.0005, 0.0002\}$. For each step size, 1000 sample paths are generated, and the plots of the mean with a 95% confidence interval are generated, along with a cumulative distribution of the final value and the margin

Table 1: Summary statistics of absolute error.

Step size, h	Mean absolute error	Standard deviation of absolute error
0.1	0.521444	0.446461
0.02	0.502813	0.421262
0.01	0.510874	0.415863
0.005	0.539743	0.418615
0.002	0.513387	0.413464
0.001	0.515804	0.420624
0.0005	0.502596	0.406794
0.0002	0.511967	0.415433

of error of the confidence interval. The mean absolute error and its standard deviation are summarized in table 1. Select plots are shown in figures 1, 2, 3, and 4. Observe that the confidence intervals increase as the simulations move forward, and finer step sizes shrink the rate at which the margin of error increases. One can determine the approximate rate by repeating the trials numerous times (perhaps 1000 runs of 1000 sample paths each). Estimates of the errors in different norms can also be made.

Stochastic differential equations serve many purposes in mathematical biology, quantum mechanics, and quantitative finance. Although the Euler-Maruyama method is simple to implement, it can give a big picture of the behavior of the true solution in expectation. As with numerical schemes for ordinary and partial differential equations, certain applications may benefit from adaptive methods, stability considerations, and extensions to systems of several equations (perhaps even with multiple Brownian motions).

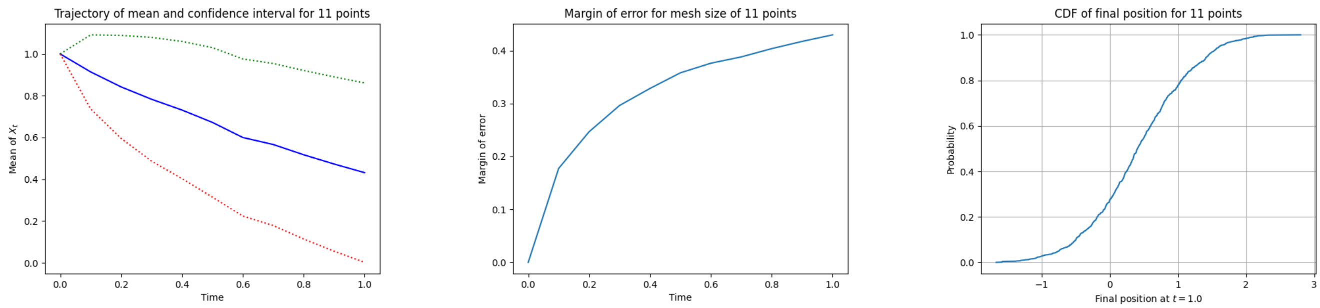

Figure 1: Plots for the step size $h = 0.1$.

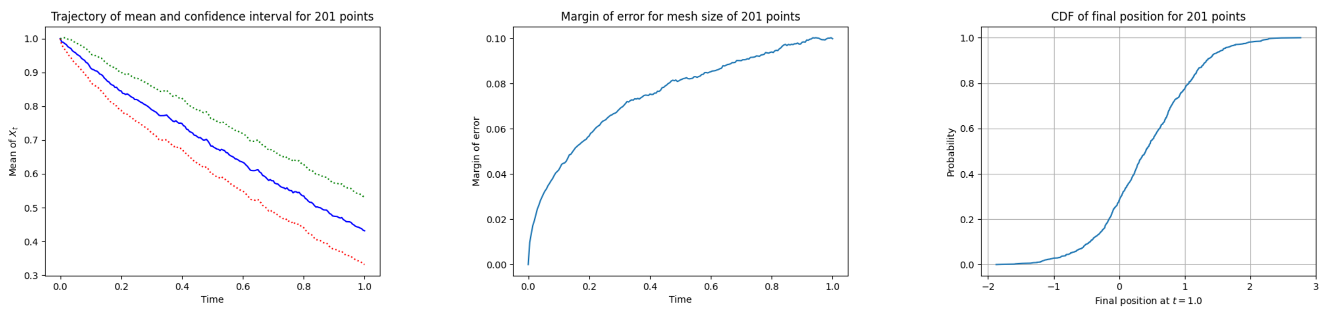

Figure 2: Plots for the step size $h = 0.005$.

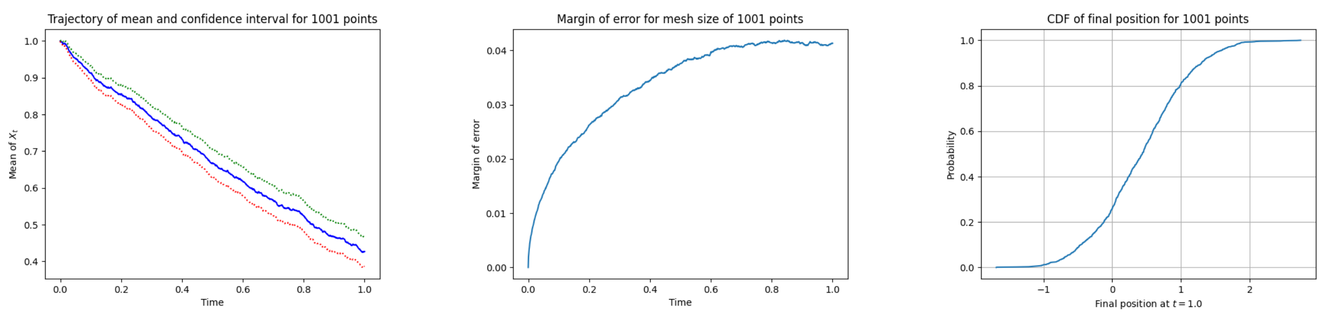

Figure 3: Plots for the step size $h = 0.001$.

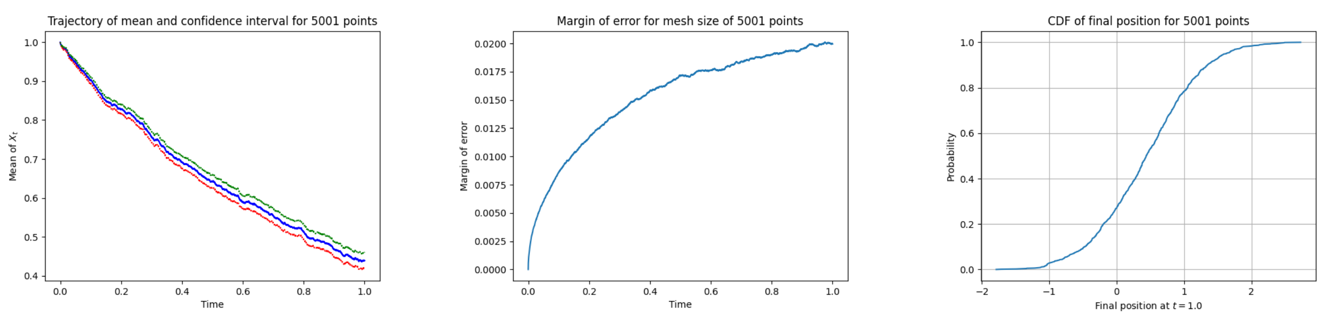

Figure 4: Plots for the step size $h = 0.0002$.

References

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[3] Gisiro Maruyama. Continuous markov processes and stochastic equations. *Rendiconti del Circolo Matematico di Palermo*, 4(1):48–90, 1955.

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[5] Sean Zachary Roberson. A quick drift into stochastic calculus. 2019.