A Quick Drift Into Stochastic Calculus

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July 30, 2023

Overview

Introduction

The stochastic calculus is a beautiful tool used in the modeling of stock prices, diffusion processes, and other random systems.
This talk will give a very brief introduction into this beautiful field.

Goals

We have four goals.

- Use the definition of the Stieltjes integral to arrive at the Ito integral.
- Demonstrate how the Ito and the Riemann integrals differ.
- Show the version of the Fundamental Theorem of Calculus for stochastic integrals.
- Solve some simple stochastic differential equations.

Flashback: The Riemann-Stieltjes Integral

Recall the Riemann-Stieltjes integral of a function f with respect to a function g of bounded variation:

$$\int_a^b f(x)\ dg(x) = \lim_{n\to\infty} \sum_{1\leq j\leq n} f(x_j)\,(g(x_j) - g(x_{j-1}))$$

where the limit is interpreted as the size of the mesh of the partition $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ goes to zero.

We define Brownian motion, called the Wiener process in some texts.

Brownian Motion

The process B_t is a Brownian motion if

1. $B_0 = 0$, a.s.
2. B_t has independent increments, i.e., the processes B_t and $B_{s+t} - B_s$ (where $s > 0$) are independent.
3. The increment given above is normally distributed with mean 0 and variance s.
4. The process B_t is a.s. continuous in t.

Build the Integral!

We'll now attempt to build the stochastic integral

$$\int_a^b f(s)\ dB_s$$

taken with respect to a Brownian motion.[1]

[1] The integrand can even be a function of a Brownian motion, but we'll consider this simple case of the integral of a deterministic function with respect to noise.

Build the Integral!

The way we do this is by replacing the function $g(x)$ in the Stieltjes integral by the Brownian motion B_s. So, in an informal sense,

$$\int_a^b f(s)\ dB_s = \lim_{n\to\infty} \sum_{1\le j\le n} f(s_j)\left(B(s_j) - B(s_{j-1})\right)$$

where the limit is actually an L^2 (mean-square) limit.

Operating with the Integral

We know that

$$\int_0^t s \; ds = \frac{t^2}{2},$$

but does

$$\int_0^t B_s \; dB_s \stackrel{?}{=} \frac{B_s^2}{2}$$

hold? We'll see that the answer is no, so the usual rules of calculus we learned before don't always hold.

An Informal Approximation

To figure out how to operate in the realm of stochastic calculus, we'll need the build the Ito Lemma.

Begin with a $\mathcal{C}^2$ function $f(t,x)$ that is differentiable in the right half-plane. We can write its Taylor expansion around the origin as usual:

$$\begin{aligned} df(t,x) = f(0,0) + f_t(0,0)\ dt + f_x(0,0)\ dx \\ + \frac{1}{2} f_{tt}(0,0)\ (dt)^2 + \frac{1}{2}\ f_{xx}(0,0)\ (dx)^2 + f_{tx}(0,0)\ dt\ dx + \dots \end{aligned}$$

We actually will only need terms up to second order.

Continuing...

Now let $x = B_t$, a Brownian motion. Then in the expansion,

$$\begin{aligned} df(t, B_t) = f(0,0) + f_t(0,0)\ dt + f_x(0,0)\ dB_t \\ + \frac{1}{2} f_{tt}(0,0)\ (dt)^2 + \frac{1}{2}\ f_{xx}(0,0)\ (dB_t)^2 + f_{tx}(0,0)\ dt\ dB_t \end{aligned}$$

Finishing Up

Now, assuming $f(0,0) = 0$, allowing $dt \to 0$, and allowing all squares (that is, dt^2, dB_t^2, and $dt\ dB_t$) to go to zero, we arrive at the following:

$$df(t, B_t) = \left(f_t + \frac{1}{2} f_{xx}\right)\ dt + f_x\ dB_t$$

The Ito Lemma

We have now arrived at the Ito Lemma.

Ito Lemma

Let $f(t,x)$ be $\mathcal{C}^2([0,\infty) \times \mathbb{R})$. Then the process $X_t = f(t, B_t)$ has the following "derivative:"

$$dX_t = \left(f_t + \frac{1}{2} f_{xx} \right) \, dt + f_x \, dB_t$$

In the integral form, X_t admits the following representation:

$$X_t = X_0 + \int_0^t \left(f_s(s, B_s) + \frac{1}{2} f_{xx}(s, B_s) \right) \, ds + \int_0^t f_x(s, B_s) \, dB_s$$

The lemma allows us to evaluate Ito integrals and determine stochastic derivatives.

You may as well call this the Fundamental Theorem of Stochastic Calculus, or the FTSC.

The Stochastic Power Rule

Let's return to the question of evaluating the integral

$$\int_0^t B_s \; dB_s.$$

This can be done by use of the Ito lemma. We begin with our guess of antiderivative and set $f(t, B_t) = \frac{B_t^2}{2}$. Then with $f(t, x) = \frac{x^2}{2}$, the lemma gives the following derivative:

$$d\left(\frac{B_t^2}{2}\right) = \frac{1}{2}\; dt + B_t \; dB_t$$

The Stochastic Power Rule

Now we integrate both sides on $[0, t]$.

$$\frac{B_t^2}{2} = \frac{1}{2}t + \int_0^t B_s \; dB_s$$

As the integral is linear, we can rearrange this to give a basic result:

$$\int_0^t B_s \; dB_s = \frac{B_t^2}{2} - \frac{1}{2}t$$

This gave us what we expected... just off by a “fudge factor.”

The Stochastic Power Rule

We can generalize this to any positive integer power.

Stochastic Power Rule

Let B_t be a Brownian motion and n a positive integer. Then

$$\int_0^t B_s^n \, dB_s = \frac{B_t^{n+1}}{n+1} - \int_0^t \frac{n}{2} B_s^{n-1} \, ds.$$

We know how to solve the equation

$$dx = k\ dt,$$

where $k \in \mathbb{R}$. Let's examine how to solve the stochastic equation

$$dX_t = k\ dt + m\ dB_t,$$

where we now have a noise term.

The Method

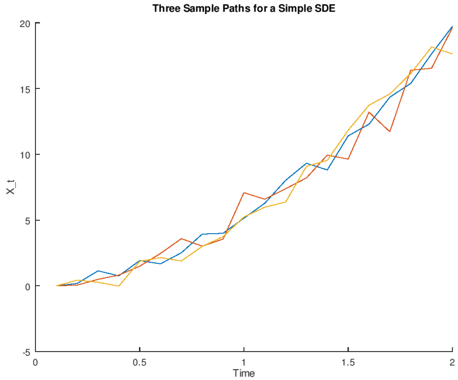
We use the Ito formula to determine the solution X_t via a system of partial differential equations. The system in play here is

$$f_x = m$$
$$f_t + \frac{1}{2} f_{xx} = k.$$

Integrating the first equation with respect to x yields $f(t, x) = mx + g(t)$, where g is a differentiable function of a single variable. We use the second equation to see that $g(t) = kt$, so $f(t, x) = mx + kt$.
The solution, then, as a stochastic process, is $\boxed{X_t = mB_t + kt}$.

A Sample Solution

Below is a plot of the solution to the equation $dX_t = dt + dB_t$ over the time interval $[0, 2]$.



Applications

There are many applications of the stochastic calculus, including:

1. Modeling prices of stocks in markets
2. Optimal stopping (best time to buy or sell)
3. Analysis of noise in PDEs
4. Fluid mechanics, thermodynamics, quantum dynamics

References

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