Analysis of a Traffic Flow Model

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Outline

- Foundational Definitions
- Conservation Equation
- Characteristics
- Examples
- Numerical Approach

Foundational Definitions

- x: location/length of road
- t: time since beginning of measurement
- $\rho(x,t)$: density at location x and time t
- $u(x,t)$: velocity of vehicles at location x and time t
- $q(x,t)$: flux/flow rate of vehicles over length of road x over time t
  - $q(x,t) = \rho(x,t)u(x,t)$

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Conservation Equation

Given the density $\rho(x,t)$ and velocity $u(x,t)$, assume that as the density of vehicles increases, the velocity of the vehicles decreases. Additionally, since $q(x,t) = \rho(x,t)u(x,t)$, we have:

$$q(x,t) = \frac{cars}{length} \cdot \frac{length}{time} = \frac{cars}{time}$$

Since cars must be conserved, the change in cars would be the cars in minus the cars out; therefore, the conservation condition on the interval $[a,b]$ is:

$$\frac{d}{dt}N(t) = q(a,t) - q(b,t)$$

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Conservation Equation

Now consider the conservation condition on the interval $[a, x]$ where x is the variable distance:

$$\frac{d}{dt}N = q(a,t) - q(x,t)$$

Then, since the integrand is continuous on, we can rewrite the conservation condition in the following way:

$$q(a,t) - q(x,t) = \frac{d}{dt}\int_a^x \rho(s,t)\,ds = \int_a^x \frac{\partial \rho(s,t)}{\partial t}\,ds$$

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Conservation Equation

If we differentiate the previous equation with respect to x, we get the following equation:

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial q(x,t)}{\partial x}$$

which we can rewrite once more as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

This gives us the conservation equation.

Conservation Equation

Since $q = \rho u$, we have:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

Then, since $u = u(\rho)$ is a decreasing function (as density increases, velocity decreases):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u(\rho)) = 0$$

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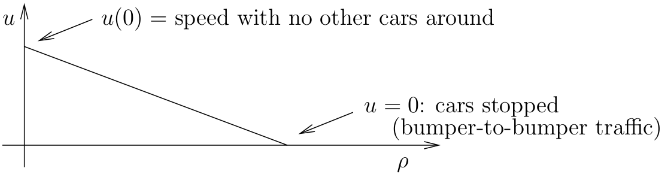
Heuristic Rule

As density of cars increase, the velocity decreases $\Rightarrow u = u(\rho)$ is a decreasing function



Then flux $= q = q(\rho) = \rho u(\rho)$ and

$$\frac{\partial q}{\partial x} = \frac{dq}{d\rho}\frac{\partial \rho}{\partial x} \quad \text{(By product rule)}$$

Hence, the conservation law is

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0$$
$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$$

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Heuristic Rule

How can we solve this?
Consider the simple case: $c(\rho) = \text{constant} \; = c$

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0$$

Looking at the solution along the line $x = x(t)$, giving a solution $\rho(x(t), t)$
Then

$$\frac{d\rho}{dt} = \rho_x \frac{dx}{dt} + \rho_t$$

So, if $\frac{dx}{dt} = c$, it follows that

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x} c + \frac{\partial \rho}{\partial t} = 0$$

along the line $x = x(t)$!

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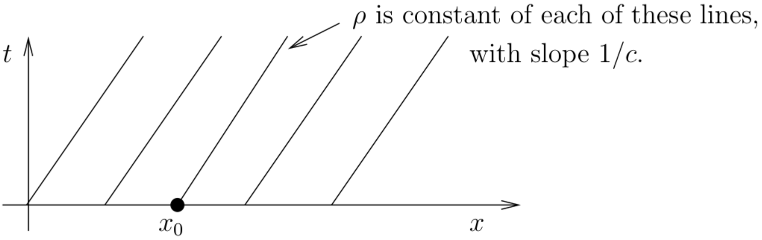
Heuristic Rule

That is, ρ is constant along any line $x(t)$ such that $x'(t) = c$, that is, along any line $x = x_0 + ct$.



These lines are called characteristics.

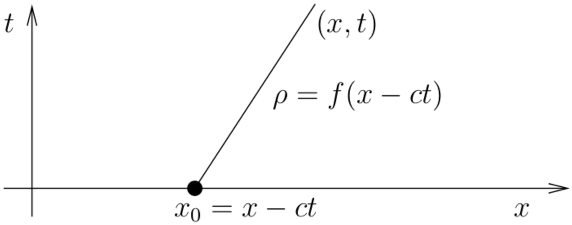
Heuristic Rule

Suppose that the initial condition is

$$\rho(x, 0) = f(x)$$

Then

$$\rho(x, t) = f(x - ct)$$



i.e., $\rho(x, t) = \rho(x_0, 0) = f(x_0) = f(x - ct)$.

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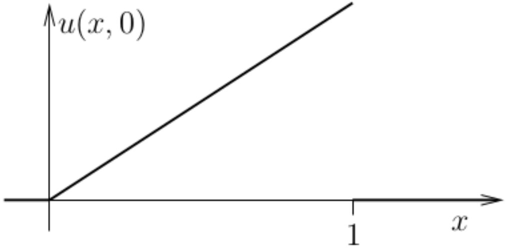
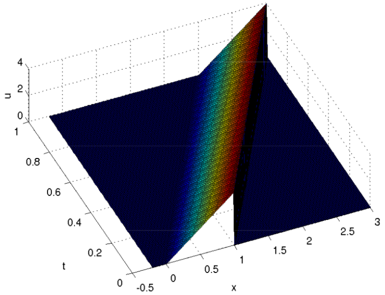
1.3.1 Example

$$u_t + 2u_x = 0$$

$$u(x,0) = \begin{cases} 0, & x < 0 \\ 4x, & 0 < x < 1 \\ 0, & x > 1 \end{cases} \Rightarrow$$



1.3.1 Example



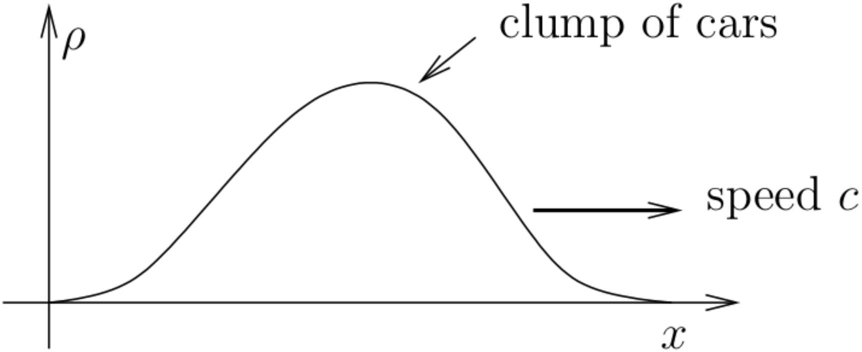
The characteristics are $x = 2t + x_0$. The hump just moves along unchanged

$$u(x,t) = \begin{cases} 4(x-2t), & 2t < x < 2t+1 \\ 0, & \text{otherwise} \end{cases}$$

Physical Interpretation

$$q'(\rho) = \text{constant} \Rightarrow q = c\rho \Rightarrow u = c \text{ (constant)}$$

All cars move at the same speed, regardless of the density.
A clump of cars remains as a clump, unchanged



Characteristics

$\rho =$ constant along the curves

$$\frac{dx}{dt} = c(\rho)$$

in general, where c is not just a constant. $c(\rho)$ is called the density wave velocity.

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Next Fact

$$\rho = \text{ constant along } \frac{dx}{dt} = c(\rho) \Rightarrow c(\rho) = \text{ constant along } = c(\rho)$$
$$\Rightarrow \text{ all characteristics are straight lines}$$

But all characteristics may have different slopes depending on the initial density

Next Fact

We require $c'(\rho) < 0$ (denser traffic must move slower)

$$\mu(\rho) = u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$$

$$q(\rho) = \rho\mu(\rho) = \rho u_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right)$$

$$q'(\rho) = c(\rho) = u_{\text{max}} \left(1 - \frac{2\rho}{\rho_{\text{max}}}\right)$$

$$c'(\rho) = \frac{-2u_{\text{max}}}{\rho_{\text{max}}} < 0$$

Observe that the maximum value of q, the maximum capacity, occurs for $c(\rho) = q'(\rho) = 0$, so for $\rho = \rho_{\text{max}}/2$ and

$$q_{\text{max}} = \frac{\rho_{\text{max}}}{2} u_{\text{max}} \left(1 - \frac{\rho_{\text{max}}/2}{\rho_{\text{max}}}\right) = \frac{\rho_{\text{max}} u_{\text{max}}}{4}$$

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Example (1.4.3)

$\rho_t + c(\rho)\rho_x = 0$, where $c'(\rho) < 0$ and

$$\rho(x,0) = \begin{cases} 4, & x < 0 \\ 3, & x > 0 \end{cases}$$

Suppose

$$\left.\begin{aligned} u_{\max} &= 1 \\ \rho_{\max} &= 10 \end{aligned}\right\} \text{ then } c(\rho) = 1 - \frac{\rho}{5}.$$

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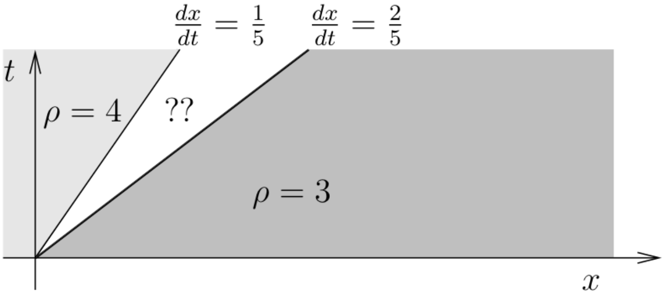
Example (1.4.3)

Characteristics are $\frac{dx}{dt} = c(\rho)$. Along these lines $\rho =$ constant.

$$\rho \Rightarrow 4 \Rightarrow \frac{dx}{dt} = 1 - \frac{4}{5} = \frac{1}{5}$$
$$\rho = 3 \Rightarrow \frac{dx}{dt} = 1 - \frac{3}{5} = \frac{2}{5}$$

The density of $\rho = 3$ moves to the right faster than that of density $\rho = 4$.



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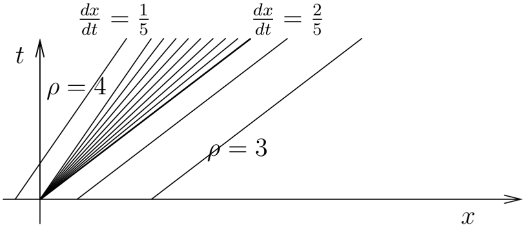
Example (1.4.3)

What happens between?

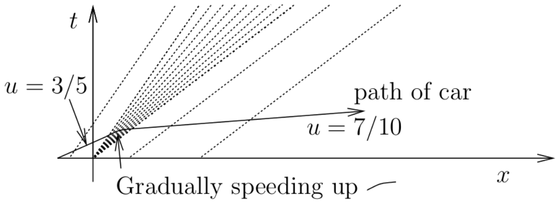
- We get characteristics with a fan shock



Example (1.4.3)

What happens between?

- Density varies smoothly from 4 to 3
- Cars initially move at speed density wave $\frac{3}{5}$
- Reaches the fan shock $\implies$ cars begins to speed up
- Speeds up $\rightarrow$ Move through the shock
- Travels at speed density $\frac{7}{10}$



Example (1.4.3)

Note:

$$\text{Speed of density wave} \neq \text{Speed of cars}$$

$$\frac{dq}{d\rho} = \frac{d}{d\rho}(\rho u(\rho)) = u + \rho u'(\rho) < u \text{ since } u'(\rho) < 0$$

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Numerical Methods

- Numerical methods can be used to estimate solutions of the PDE.
- As we are considering hyperbolic equations, we must be careful about stability.
- For the moment, we discuss a simple finite difference scheme and finite volumes.

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Finite Differences

We first begin by discretizing our continuous problem. For the case of constant propagation speed, we consider the differential equation

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0$$

subject to the initial condition $\rho(x, 0) = f(x)$. We let R_i^j be the approximated solution at x_i, t_j, with grid sizes h and k respectively. Then a finite difference scheme

$$R_i^{j+1} = R_i^j - \frac{ck}{h}\left(R_i^j - R_{i-1}^j\right)$$

provides an iterative process to solve the problem.

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Stability Concerns, Accuracy

In order to ensure stability, we require the following CFL condition (named for Courant, Friedrichs, and Lewy):

$$\left|\frac{ck}{h}\right| \leq 1.$$

Such a scheme is first-order accurate, which can be proved via Taylor expansions.

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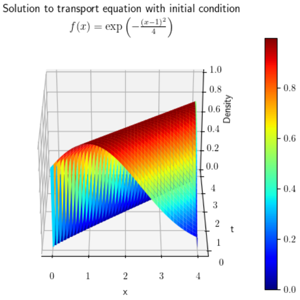
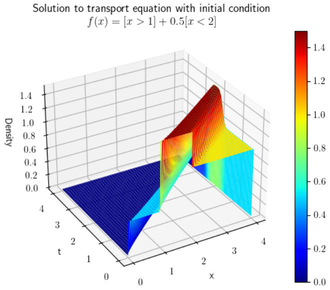
A First Example

Both examples below are solutions to the equation $\rho_t + 1.25\rho_x = 0$ with different initial conditions.



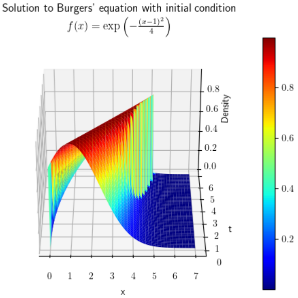
(a) Solution with Gaussian initial data.



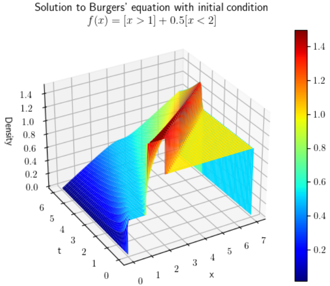
(b) Solution with a step function as initial profile.

A Test with Burgers' Equation

With the same initial conditions as before, we now use finite differences for Burgers' equation $\rho_t + \rho\rho_x = 0$.



(a) Solution with Gaussian initial data.



(b) Solution with a step function as initial profile.

The Godunov Method

To handle discontinuous initial conditions (and solutions) we use a modified method based on weak solutions.

Weak Solution

Let L be a differential operator. The function u is a weak solution to the differential equation $L(u)(x,t) = f(x,t)$ if, for every test function $v \in C^1(X)$ with compact support,

$$\int_X L(u)(x,t)v(x,t)\ d(x,t) = \int_X f(x,t)v(x,t)\ d(x,t).$$

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Establishing the Scheme

For the transport equation, the Godunov scheme will do the following:

- Create a space-time grid. Given the spatial grid $\{x_j\}$, define the volume $V_j = \{x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\}$ where $x_{j+\frac{1}{2}}$ is the midpoint of the interval (x_j, x_{j+1}).
- Solve the differential equation on each volume V_j so that the solutions are piecewise constant. (This is a Riemann problem.)
- Repeat for each point in the time grid.

Other solution types can be used (piecewise linear, e.g.) but the computation is more involved. This finite volume method (much like the finite element method for elliptic equations) is based on the integral formulation.

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An Example Solution

Once again, we solve Burgers' equation with a Gaussian initial condition with the change so that the initial density is non-zero in a certain interval.

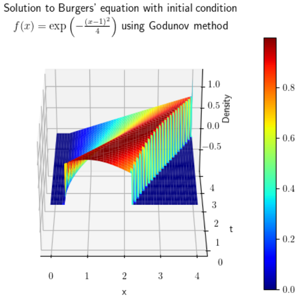

Figure: Solution with Gaussian initial data.