Vector Calculus and Differential Forms with Applications to Electromagnetism

Sean Roberson

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PREFACE

This paper is written as a final project for a course in vector analysis, taught at Texas A&M University - San Antonio in the spring of 2015 as an independent study course.

Students in mathematics, physics, engineering, and the sciences usually go through a sequence of three calculus courses before going on to differential equations, real analysis, and linear algebra. In the third course, traditionally reserved for multivariable calculus, students usually learn how to differentiate functions of several variable and integrate over general domains in space. Very rarely, as was my case, will professors have time to cover the important integral theorems using vector functions: Green's Theorem, Stokes' Theorem, etc. In some universities, such as UCSD and Cornell, honors students are able to take an accelerated calculus sequence using the text *Vector Calculus, Linear Algebra, and Differential Forms* by John Hamal Hubbard and Barbara Burke Hubbard. Here, students learn multivariable calculus using linear algebra and real analysis, and then they generalize familiar integral theorems using the language of differential forms. This paper was written over the course of one semester, where the majority of the book was covered.

Some details, such as orientation of manifolds, topology, and the foundation of the integral were skipped to save length. The paper should still be readable by a student with at least three semesters of calculus, one course in linear algebra, and one course in real analysis - all at the undergraduate level. Many of the fundamental theorems do not have their proofs, for some would be lengthy or require many details from real analysis.

The idea of this paper is to introduce differential forms - an object that allows Maxwell's equations of electromagnetism to be concisely represented in two lines as opposed to four. The development of this idea takes time, but it is worth it in the long run.

-Sean Zachary Roberson

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FUNDAMENTAL DIFFERENTIAL OPERATORS

In vector analysis, it is of interest to examine the rate of flow of various quantities such as heat, electricity, and fluids. The concept of derivative can be extended to vectors in three ways, each using some sort of partial derivative. Each differential operator uses the symbol ∇ to operate on a function of several variables. This symbol is treated as the following vector:

$$\nabla := \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$$

1.1 GRADIENT

The first differential operator is the gradient, denoted by ∇f or $\operatorname{grad} f$. The gradient operates on a $\mathcal{C}^1$ function $f : \mathbb{R}^n \to \mathbb{R}$ and returns a vector of first derivatives. For example, if $f(x, y, z) = 2x + 3xy - 4yz^2$, then

$$\nabla f = \begin{pmatrix} 2 + 3y \\ 3x - 4z^2 \\ -8yz \end{pmatrix}$$

There are scalar product and quotient rules for gradients, with

$$\operatorname{grad}(fg) = f \operatorname{grad} g + g \operatorname{grad} f$$

and

$$\operatorname{grad}\left(\frac{f}{g}\right) = \frac{g \operatorname{grad} f - f \operatorname{grad} g}{g^2},$$

similar to the product and quotient rules familiar in single variable calculus.

1.2 DIVERGENCE

The next operator is the divergence, and it returns a sum of partial derivatives of a vector function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$. Using the nabla notation, divergence can be denoted by $\nabla \cdot \mathbf{F}$, or as div $\mathbf{F}$. By definition,

$$\operatorname{div} \mathbf{F} := \sum_{1 \leq j \leq n} \frac{\partial \mathbf{F}_j}{\partial x_j}.$$

Since div is a differential operator, there are product and quotient rules for differentiation. The product rule states that for a scalar function f and a vector function $\mathbf{G}$,

$$\operatorname{div}(f\mathbf{G}) = \mathbf{G} \cdot \operatorname{grad} f + f \operatorname{div} \mathbf{G}.$$

Similarly, the quotient rule states

$$\operatorname{div}\left(\frac{\mathbf{G}}{f}\right) = \frac{f \operatorname{div} \mathbf{G} - \mathbf{G} \cdot \operatorname{grad} f}{f^2}.$$

Another differential operator can be built from the divergence operator. The *Laplacian* is a differential operator that returns a sum of second derivatives. It it defined as

$$\nabla^2 := \nabla \cdot \nabla,$$

or, when acting on a function f,

$$\begin{aligned}
\nabla^2 f &= \operatorname{div} \operatorname{grad} f \\
&= \nabla \cdot \nabla f \\
&= \sum_{1 \leq j \leq n} \frac{\partial^2 f}{\partial x_j^2}.
\end{aligned}$$

The Laplacian appears in many partial differential equations, such as the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

the *diffusion equation*

$$\frac{\partial u}{\partial t} = k \nabla^2 u,$$

and the *Laplace equation* $\nabla^2 u = 0$. It is important to note that the Laplacian operator acts on spatial variables such as x, y, z and not the time variable t (Strauss, 14).

1.3 CURL

The last fundamental differential operator exists only in $\mathbb{R}^3$. The curl returns a vector of derivatives according to the rule $\nabla \times \mathbf{F}$. That is,

$$\operatorname{curl} \mathbf{F} := \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}.$$

The curl of a product of a scalar and vector function is given by

$$\operatorname{curl}(f\mathbf{G} = f \operatorname{curl} \mathbf{G} + \operatorname{grad} f \times \mathbf{G},$$

and the curl of the quotient of a vector function with a scalar function is

$$\operatorname{curl}\left(\frac{\mathbf{G}}{f}\right) = \frac{f \operatorname{curl} \mathbf{G} - \mathbf{G} \times (\operatorname{grad} f)}{f^2}.$$

We also have two important results regarding the curl.

Theorem 1. *Suppose f is differentiable. Then* $\operatorname{curl} \operatorname{grad} f = \mathbf{0}$.

Proof. By definition,

$$\operatorname{curl} \operatorname{grad} f = \begin{pmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{pmatrix}$$

$$= \mathbf{0}$$

since mixed partials commute. $\square$

Theorem 2. *Suppose $\mathbf{F}$ is a differentiable vector function on $\mathbb{R}^3$. Then* $\operatorname{div} \operatorname{curl} \mathbf{F} = 0$.

Proof. By definition,

$$\operatorname{div} \operatorname{curl} \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix}^T \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0,$$

since, as above, mixed partials commute. $\square$

There are many other product rules for vector derivatives. Griffiths gives the following three product rules utilizing both the inner and cross products:

1. $\text{grad}(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \times \text{curl}\,\mathbf{G}) + (\mathbf{G} \times \text{curl}\,\mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F}$

2. $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl}\,\mathbf{F} - \mathbf{F} \cdot \text{curl}\,\mathbf{G}$

3. $\text{curl}(\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\text{div}\,\mathbf{G}) - \mathbf{G}(\text{div}\,\mathbf{F})$

CLASSICAL THEOREMS AND RESULTS

As with any branch of mathematics, vector calculus has its own set of classic results. The theorems here are presented in the order in which they appear in James Stewart's *Calculus: Early Transcendentals*.

2.1 FUNDAMENTAL THEOREM FOR LINE INTEGRALS

The Fundamental Theorem for Line Integrals, also called the Gradient Theorem by David Griffiths gives a way to evaluate special line integrals (p. 29). In order to understand the theorem, we must first define a *conservative vector field* and a *potential function.*

Definition 1 (Conservative Vector Field, Potential Function)**.** *A vector field is conservative if it is the gradient of some differentiable function. That is,* **F** *is conservative iff* **F** $= \operatorname{grad} f$. *Such a function* f *is called a potential function for* **F**.

Finding these potential functions amounts to solving an exact differential equation, defined below, along with the definition of an exact differential.

Definition 2 (Exact Differential, Exact Differential Equation)**.** *Let* $z = f(x,y)$ *be a differentiable function. The symbol dz denotes the differential of f, given as $\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$. The expression $M\,dx + N\,dy$ is an exact differential if $M = \frac{\partial f}{\partial x}$ and $N = \frac{\partial f}{\partial y}$ - that is, it corresponds to the differential of a function. The associated differential equation $M\,dx + N\,dy = 0$ is exact if the left side is an exact differential.*

Solving these exact differential equations uses the equality of mixed derivatives. This establishes the following exactness condition; a similar expression will be used in the computations for Green's Theorem.

Theorem 3 (Condition for Exactness)**.** *A necessary and sufficient condition for $M\,dx + N\,dy$ to an exact differential on some rectangle R is*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

provided that M and N are differentiable on R.

The proof for the necessity is given in Zill, while the proof of the sufficiency outlines the method of solution to an exact differential equation (p. 64). To solve such an equation, one of the functions M or N is selected and then integrated with respect to their "attached" variable - that is, integrate M with respect to x, and N with respect to y. Afterwards, differentiate with respect to the remaining variable and then solve for $f(x,y)$. This procedure is similar to finding a potential function for a conservative vector field.

Theorem 4 (Conservative Criterion). *If* $\mathbf{F} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ *is conservative, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

The converse holds on an open, simply-connected region - a region whose boundary curve does not intersect itself and is convex.

Thus, to find a potential function for the vector field $\mathbf{F}$ described above, it suffices to solve the exact differential equation

$$P \, dx + Q \, dy = 0.$$

We are now ready to state the Fundamental Theorem of Line Integrals.

Theorem 5 (Fundamental Theorem of Line Integrals, Gradient Theorem). *Let C be a smooth curve parameterized by* $\mathbf{r}(t), a \leq t \leq b$. *Suppose f is differentiable with a continuous gradient on C. Then*

$$\int_C \operatorname{grad} f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

From the Gradient Theorem follows a property of integrals taken over a closed loop.

Theorem 6 (Path Independence Theorem). *The integral* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is path independent on a domain D iff* $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for each closed path* $C \subset D$.

2.2 GREEN'S THEOREM

The following result is due to George Green. It relates the line integral of a function over a closed curved to an associated double integral taken over the interior of the curve.

Theorem 7 (Green's Theorem). *Suppose C is a curve with positive orientation (that is, oriented in the counterclockwise direction). Suppose also that C is smooth and closed. If P and Q are of class C^1 on some open set contained in $\mathring{C}$, then*

$$\int_C P \, dx + Q \, dy = \iint_{\mathring{C}} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA.$$

It immediately follows from the theorem that if the integrand on the left is an exact differential, the integral is zero.

We may also write Green's Theorem in two vector forms: one with the curl and the other with the divergence.

Theorem 8 (Vector Forms of Green's Theorem). *Suppose $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field in $\mathbb{R}^3$ whose component functions are of class C^1, and suppose C is a simple, smooth, closed curve. Then*

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathring{C}} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} \, dA.$$

Equivalently, if $\mathbf{n}$ is an outward-oriented normal to the curve C, then

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_{\mathring{C}} \nabla \cdot \mathbf{F} \, dA.$$

From the vector forms of Green's theorem, two identities follows. The first can be viewed as a generalized form of integration by parts.

Corollary 1 (Green's First Identity). *Suppose f is of class C^1 and g is of class C^2, and that the compact set D and its boundary satisfy Green's Theorem. Then*

$$\iint_D f \nabla^2 g \, dA = \oint_{\partial D} f(\nabla g) \cdot \mathbf{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA.$$

In order to prove the identity, the following property about divergence is needed.

Lemma 1. *Suppose f is $\mathcal{C}^1$ and g is $\mathcal{C}^2$. Then $\nabla \cdot (f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$.*

Proof of Lemma. We have

$$\nabla \cdot (f\nabla g) = \frac{\partial f g_x}{\partial x} + \frac{\partial f g_y}{\partial y} + \frac{\partial f g_z}{\partial z}$$
$$= f g_{xx} + f_x g_x + f g_{yy} + f_y g_y + f g_{zz} + f_z g_z$$
$$= f\nabla^2 g + \nabla f \cdot \nabla g.$$

$\square$

We are now ready to prove Green's First Identity.

Proof of Green's First Identity. Begin with the integral of $f\nabla g \cdot n$ taken over the boundary of D. By the lemma and the second vector form of Green's Theorem,

$$\oint_{\partial D} f\nabla g \cdot n \, ds = \iint_D \nabla \cdot (f\nabla g) \, dA$$
$$= \iint_D f\nabla^2 g + \nabla f \cdot \nabla g \, dA.$$

Rearranging yields the desired.

$\square$

Corollary 2 (Green's Second Identity). *Suppose f and g satisfy the hypotheses of the First Identity. Then*

$$\iint_D f\nabla^2 g - g\nabla^2 f \, dA = \oint_{\partial D} (f\nabla g - g\nabla f) \cdot \mathbf{n} \, ds.$$

Proof. Apply the First Identity to the integral on the left, separating the integrand. Then the only term that remains is the line integral.

$\square$

Using Green's identities, we can examine what happens to functions that are harmonic on a set. That is, if a function f satisfies Laplace's equation:

$$\nabla^2 f = 0,$$

then the following theorem holds.

Corollary 3. *Suppose D is a region that satisfies Green's theorem and g is harmonic on D. Then $\oint_{\partial D} \nabla g \cdot \mathbf{n} \, ds = 0$.*

Proof. Let $f(x, y, z) = 1$. By the First Identity,

$$\iint_D f \nabla^2 g \, dA = \oint_{\partial D} \nabla g \cdot \mathbf{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA$$

$$= \oint_{\partial D} \nabla g \cdot \mathbf{n} \, ds - \iint_D \mathbf{0} \cdot \nabla g \, dA$$

$$= \oint_{\partial D} \nabla g \cdot \mathbf{n} \, ds$$

Now, since g is harmonic on D, the first integral is zero. Hence the result holds. $\square$

Corollary 4. *Suppose f is harmonic on a region D satisfying Green's Theorem and $f \equiv 0$ on ∂D. Then $\iint_D |\nabla f|^2 \, dA = 0$.*

Proof. Set $f = g$ in the First Identity. Then

$$\iint_D f \nabla^2 f \, dA = \oint_{\partial D} f \nabla f \cdot \mathbf{n} \, ds - \iint_D |\nabla f|^2 \, dA$$

$$= -\iint_D |\nabla f|^2 \, dA.$$

But $\nabla^2 f = 0$ on D, so the integral on the left is zero. Also, note that the norm functions is non-negative, and the only way $|\nabla f|^2$ can be zero is if $\nabla f = 0$. Hence, the entire integral is zero (and, consequently, f is identically zero on the entire domain D). $\square$

2.3 STOKES' THEOREM

The next theorem is due to George Stokes. As another integral theorem, it relates a line integral to a surface integral under appropriate conditions. The generalization of Stokes' Theorem will be explored once differential forms and exterior derivatives are introduced.

Theorem 9 (Stokes' Theorem)**.** *Let S be a surface whose boundary curve is simple, closed, and has positive orientation. Suppose that the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ has continuous partial derivatives on an open set containing S. Then*

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \, \mathbf{F} \cdot \mathbf{n} \, dS,$$

> *Where the integral on the right is a surface integral.*

As with Green's Theorem, Stokes' Theorem changes difficult integrals into easier problems.

2.4 GAUSS' DIVERGENCE THEOREM

In mathematics, it is common for the same result to be discovered independently by different mathematicians. For example, the Cauchy-Schwarz inequality for sums was discovered by Augustin-Louis Cauchy, while the related inequality for integrals was given by Viktor Bunyakovsky. The next theorem is credited to Carl Gauss, while Mikhail Ostrogradsky published the result in 1826 (Stewart, p.1129).

Theorem 10 (Divergence Theorem)**.** *Suppose E is a solid and S is the surface (or union of surfaces), with outward orientation, that encloses E. Let **F** be a $\mathcal{C}^1$ vector field on an open set containing E. Then*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \nabla \cdot \mathbf{F} \, dV.$$

Corollary 5 (Volume Formula)**.** *Suppose E is a solid satisfying the hypothesis of the Divergence Theorem. Then the volume of E is given by $\frac{1}{3} \iint_S (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot d\mathbf{S}$.*

Proof. Note that $\nabla \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 3$. The result follows. $\square$

3

DIFFERENTIAL FORMS

It is common in mathematics to find multiple representations for the same object or quantity. Such methods are useful in vector calculus. In the typical Euclidean vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, results in multivariable calculus involving areas and volumes are easy to write with integrals, but generalized results in $\mathbb{R}^n$ may be cumbersome to write down. The use of differential forms (shortened to forms) allows these results to be streamlined using a generalized differential operator while maintaining traditional methods of integration.

3.1 FORMS

The differential form is a function that acts on a given set of vectors and returns a number. There are three defining properties of forms:

1. Forms are antisymmetric. That is, exchanging any two vectors changes the sign of the result.

2. Forms are normalizing. That is, when a form acts on the standard basis vectors, it returns 1.

3. Forms are multilinear. That is, a linear combination of vectors can be separated in the form.

Notice that the properties of forms are similar to those for determinants. Thus the determinant is an example of a form on $\mathbb{R}^n$.

A form is classified by the number of vectors it acts on and the vector space it lives in. For example, a 2-form on $\mathbb{R}^3$ can look like $2x + y - z \, dx \wedge dz$, and a 3-form on $\mathbb{R}^6$ can look like $x_1 x_2 \, dx_2 \wedge dx_5 \wedge dx_6$. The wedges determine how to act on the vectors provided (the example forms given above do not have specific vectors). For now, we can focus on the wedge product and not the function at the front of the product. To see this, suppose φ is a k-form on $\mathbb{R}^n$. In particular, let $\varphi = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a set of vectors in $\mathbb{R}^n$. Build the matrix A whose columns are each $\mathbf{v}_j$. Then φ takes, in succession, each row $i_1, i_2, \ldots i_k$ from A, builds a new matrix B, then takes the determinant of B. Notice that if $i_m = i_n$, then φ returns the value zero.

3.2 FORMS AND INTEGRALS

Students in traditional multivariable calculus courses see differential forms, but do not use them in the context of vector analysis and exterior algebra. The exposure of differential forms is usually restricted to line integrals in these courses. For example, what is the value of the integral $\int_C x\,dy$, where C is the quarter-circle joining $(2,0)$ and $(0,2)$? In a traditional class, the path C is given a parameterization $\gamma(t)$, then the derivative of γ is taken. From there, the students substitute the appropriate expressions and evaluate an integral in t. Using the standard parameterization $\gamma(t) = (2\cos t, 2\sin t)$, the computation is as follows:

$$
\begin{aligned}
\int_C x\,dy &= \int_{[0,\frac{\pi}{2}]} (2\cos t)\,d(2\sin t) \\
&= \int_0^2 4\cos^2 t\,dt \\
&= \pi
\end{aligned}
$$

The computation is the same using differential forms, except we must now consider the 1-form $x\,dy$ as a function that acts on the vector $(-2\sin t, 2\cos t)$. We then replace x with $2\cos t$, as in the parameterization γ, then replace dy with $2\cos t\,dt$. The second replacement occurs from the y in the differential dy that selects the second entry in the vector. The computation of the integral remains the same.

We can extend this process to "larger" forms. These integrals of forms have connections to standard integrals in vector calculus. For instance, what is the integral of the 2-form $x\,dy \wedge dz$, over the surface parametrized by $\gamma(u,v) = (u^2, u+v, v^3)$, where $(u,v) \in S = [-1,1]^2$? To start, we must first examine the derivative of the parametrization γ. These derivatives can be placed in a 3×2 matrix, where each column represents a partial derivative with respect to each parameter and each row represents the gradient of the function in a particular entry. The derivative is then

$$
[\mathbf{D}\gamma] = \begin{pmatrix} 2u & 0 \\ 1 & 1 \\ 0 & 3v^2 \end{pmatrix}.
$$

We use this derivative to evaluate $\int_{[\gamma(S)]} x\,dy \wedge dz$. First, we examine the parametrization and replace x with u^2. Now, the wedge product $dy \wedge dz$ says to take the second and third rows of the derivative matrix and take the determinant. Call this matrix B, so

$$
B = \begin{pmatrix} 1 & 1 \\ 0 & 3v^2 \end{pmatrix},
$$

and hence $\det B = 3v^2$. The last step is to write the integral as an ordinary double integral, making appropriate choices for the limits of integration (the limits are given in the constraint for S). So,

$$\int_{[\gamma(S)]} x \, dy \wedge dz = \int_{-1}^{1} \int_{-1}^{1} 3u^2 v^2 \, du \, dv$$

$$= 3 \left(\int_{-1}^{1} w^2 \, dw \right)^2$$

$$= 3 \cdot \frac{2}{3}$$

$$= 2.$$

It is important to note that integrals of 2-forms are analogous to surface integrals in traditional multivariable calculus courses. In the traditional manner, we may express the integral as

$$\iint_S x \, d\sigma,$$

where σ is a surface area element of S.

3.3 THE EXTERIOR DERIVATIVE

The derivative has many interpretations in calculus. It usually represents slopes of curves, but can be generalized to represent a rate of change of a certain quantity. For functions of several variables, the partial derivatives examine change in one variable while others are held fixed. Directional derivatives give change in the direction of a particular unit vector, usually combining partial derivatives. The gradient is a special differential operator that returns a vector of partial derivatives. This vector points in the direction of greatest change.

In vector analysis (and in some cases, tensor analysis), a derivative is needed for integrals (such as in Stokes' Theorem) and relating various forms. To achieve this goal, the exterior derivative is needed. Hubbard and Hubbard define the exterior derivative as follows:

Definition 3. *Let U be an open subset of $\mathbb{R}^n$ and φ a k+1-form on U. The exterior derivative $\mathbf{d}\varphi$ is given by:*

$$\mathbf{d}\varphi(P_{\mathbf{x}}(\mathbf{v}_1, \ldots, \mathbf{v}_{k+1})) = \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial P_{\mathbf{x}}(h\mathbf{v}_1, \ldots, h\mathbf{v}_{k+1})} \varphi,$$

where $P_{\mathbf{x}}(\cdot)$ represents a parallelogram in k dimensions spanned by the vectors inside the parentheses and with a vertex at $\mathbf{x}$.

This definition is motivated by replacing the ordinary difference quotient with an integral. For functions of one variable, the derivative of f is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The expression $f(x+h) - f(x)$ can be replaced by the integral $\int_{\partial P_x(h)} f$, which says to evaluate f at $x + h$, then subtract the value of f at x. Notice how there is no mention of antiderivatives. When using this formulation for the exterior derivative of differential forms, one is typically not interested in antiderivatives, since integrals of forms can be converted to standard multiple integrals.

As with the definition of the derivative in one variable, using the definition to compute an exterior derivative is cumbersome. There are rules used to find exterior derivatives.

1. The operator $\mathbf{d}$ is linear.

2. If φ is a constant form, then $\mathbf{d}\varphi = 0$.

3. The exterior derivative of a 0-form (a function) f is its total differential:

$$\mathbf{d}f = \sum_{1 \le j \le n} \frac{\partial f}{\partial x_j} \, dx_j$$

4. If f is a function, then $\mathbf{d}\left(f \bigwedge_{1 \le j \le k} dx_{i_j} \right) = \mathbf{d}f \wedge \bigwedge_{1 \le j \le k} dx_{i_j}$.

To see how the exterior derivative works, first consider the function of two variables $f(x, y) = 3\ln xy - 2\sinh x$. Then by the third rule, $\mathbf{d}f = df$, that is,

$$\mathbf{d}f = \frac{\partial}{\partial x} (3\ln xy - 2\sinh x) \, dx + \frac{\partial}{\partial y} (3\ln xy - 2\sinh x) \, dy$$
$$= \frac{3}{x} - 2\cosh x \, dx + \frac{3}{y} \, dy.$$

As another example, let $\varphi = 2xz^2 \, dx \wedge dy - 3x^3 dy \wedge dz$. Then

$$\mathbf{d}\varphi = \mathbf{d}(2xz^2) \wedge dx \wedge dy - \mathbf{d}(3x^3) \wedge dy \wedge dz$$
$$= (2z^2 \, dx + 4xz \, dz) \wedge dx \wedge dy - 9x^2 dx \wedge dy \wedge dz$$
$$= 2z^2 \, dx \wedge dx \wedge dy + 4xz \, dz \wedge dx \wedge dy - 9x^2 \, dx \wedge dy \wedge dz$$
$$= 4xz \, dx \wedge dy \wedge dz - 9x^2 \, dx \wedge dy \wedge dz$$
$$= (4xz - 9x^2) \, dx \wedge dy \wedge dz.$$

Perhaps an interesting property of the exterior derivative is that second exterior derivatives are always zero.

> **Theorem 11.** *Suppose φ is a k-form of class C^2. Then $\mathbf{d}(\mathbf{d}\varphi) = 0$.*

Proof. If φ is a 0-form (namely a function f), then

$$
\begin{aligned}
\mathbf{d}(\mathbf{d}\varphi) &= \mathbf{d}\left(\sum_{1 \le j \le n} \frac{\partial f}{\partial x_j}\, dx_j\right) \\
&= \sum_{1 \le j \le n} \mathbf{d}\left(\frac{\partial f}{\partial x_j}\, dx_j\right) \\
&= \sum_{1 \le j \le n} \mathbf{d}\left(\frac{\partial f}{\partial x_j} \wedge dx_j\right) \\
&= \sum_{1 \le i,j \le n} \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\, dx_j \wedge dx_i\right) \\
&= 0.
\end{aligned}
$$

Otherwise, if φ is a k-form, use the preceding to obtain the result. $\square$

There is also a product rule for forms.

> **Theorem 12** (Wedge Product Rule)**.** *Suppose φ is a k-form and ψ is an l-form. Then $\mathbf{d}(\varphi \wedge \psi) = \mathbf{d}\varphi \wedge \psi + (-1)^k \varphi \wedge \mathbf{d}\psi$.*

Note that the product rule does not depend on l.

3.4 SOME BASIC FORMS: WORK, FLUX, AND MASS

There are three kinds of forms that will characterize the integral theorems. They come from their interpretations in physics and engineering. Also covered in Hubbard and Hubbard is a 0-form field, which is simply a function that returns a real number.

3.4.1 *Work*

In physics, the work done by a force is the product of that force's magnitude and the distance the object moves. With this in mind, we define the following form.

> **Definition 4** (Work Form)**.** *Let $\mathbf{F}$ be a vector field in $\mathbb{R}^n$. The work form $W_{\mathbf{F}}$ of $\mathbf{F}$ is defined as*
>
> $$
> W_{\mathbf{F}} := \sum_{1 \le j \le n} F_j\, dx_j.
> $$

For example, a work form on $\mathbb{R}^2$ looks like $2x\,dx + 3y\,dy$ and one on $\mathbb{R}^4$ looks like $\sin t\,dt + \cos 2xt\,dx - 3yz\,dy - 2tz^2\,dz$. Work forms do not need every variable present; for example, the force of gravity has the work form $-gm\,dz$ when associated to the vector field $\mathbf{F}(x, y, z) = -gm\mathbf{e}_3$ (where $\mathbf{e}_3$ is the standard basis vector associated with the z direction in $\mathbb{R}^3$).

If we wish to compute the work done by a vector field over a path, we can use a line integral.

Definition 5 (Work Integral). *The work done by a vector field $\mathbf{F}$ over an oriented curve C is the line integral*

$$\int_C W_{\mathbf{F}}.$$

Work is measured in units of energy, such as the Joule or the erg. The vector field associated to a work form gives a measure of energy per unit length.

3.4.2 *Flux*

Flux is the measure of flow through an object. The term is commonly used in electromagnetism to describe the flow of electric field lines through a surface (Serway, p. 673). Gauss' law gives a way to compute flux. Here we define a differential form used in computing flux for vector fields in $\mathbb{R}^3$.

Definition 6 (Flux Form). *A flux form $\Phi_{\mathbf{F}}$ is the 2-form field*

$$F_1\,dx \wedge dy - F_2\,dx \wedge dz + F_3\,dy \wedge dz.$$

The minus sign on the F_2 term is necessary since it arises when computing the determinant $\det[\mathbf{F}, \mathbf{v}, \mathbf{w}]$ where $\mathbf{v}$ and $\mathbf{w}$ are the vectors (usually derivatives) that the flux form acts on.

Computing the flux through a surface is similar to computing work.

Definition 7 (Flux Integral). *The flux of a vector field $\mathbf{F}$ through the surface S is the surface integral*

$$\int_S \Phi_{\mathbf{F}}.$$

Flux is measured in mass per unit area.

3.4.3 *Mass*

The last form that will be used in the integral theorems is the mass form.

Definition 8. *The mass form of a function f defined on a set $U \subset \mathbb{R}^3$ is defined by*

$$M_f := f \, dx \wedge dy \wedge dz.$$

This comes from the expression $f(\mathbf{x}) \, \det[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, where $\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are the vectors the form acts on, usually derivatives.

We can compute masses of objects given their densities.

Definition 9 (Mass Integral)**.** *The mass of an object with density function f is*

$$\int_{[\gamma(U)]} M_f.$$

The mass form is measured in mass per unit volume, or in the case of electrostatics, charge per unit volume.

3.4.4 *Derivatives*

To obtain a different version of Maxwell's equations using forms later, we need the following facts for functions and vector fields on $\mathbb{R}^3$ and the derivatives of the associated forms.

1. $\mathbf{d}f = W_{\nabla \mathbf{F}}$

2. $\mathbf{d}W_{\mathbf{F}} = \Phi_{\nabla \times \mathbf{F}}$

3. $\mathbf{d}\Phi_{\mathbf{F}} = M_{\nabla \cdot \mathbf{F}}$

So, the exterior derivative takes functions to work forms of gradients, work forms to flux forms of curls, and flux forms to mass forms of divergences.

3.5 INTEGRAL THEOREMS IN THE LANGUAGE OF FORMS

All the classical results in vector calculus can be cast in the language of forms. In particular, the integral theorems can be expressed using forms, usually in a more concise manner. The results presented here are given in the order in which they appear in Hubbard's *Vector Calculus, Linear Algebra, and Differential Forms.*

3.5.1 *Stokes' Theorem, Generalized*

All of the integral theorems in calculus can be summarized with the most general form of Stokes' Theorem. The general statement requires knowledge of manifolds - a generalization of curves and surfaces to higher dimensions. Part of the general theorem mentions a "compact piece-with-boundary." A piece-with-boundary of a manifold M is a compact set $X \subset M$ - a set that is both closed and bounded - such that the set of non-smooth points of the boundary of X in M has volume zero, and the smooth boundary has finite volume.

> **Theorem 13** (Generalized Stokes' Theorem). *Let X be a compact piece-with-boundary of an oriented manifold M. Give ∂X the boundary orientation - either "clockwise" (direct) or "counterclockwise" (indirect). Let φ be a $(k-1)$-form defined on an open set containing X. Then*
>
> $$\int_{\partial X} \varphi = \int_X \mathbf{d}\varphi.$$

Stokes' Theorem says that the integral of a form taken over the boundary of a manifold is equal to the integral of the exterior derivative of the form taken over the manifold. Compare this to the one-variable Fundamental Theorem of Calculus:

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where the right side is an "integral" over the boundary of the interval $[a, b]$: a 1-dimensional manifold in $\mathbb{R}$. A general proof of Stokes' Theorem is difficult to write: Hubbard and Hubbard give an informal proof that relies on estimating the integral using parallelograms in n dimensions; a much more generalized proof is outlined in the appendix of their text. Their generalized proof requires a solid foundation in real analysis, linear algebra, and the theory of calculus on manifolds.

3.5.2 *The Fundamental Theorem for Line Integrals*

We now express the Gradient Theorem without using the word "gradient."

> **Theorem 14** (Fundamental Theorem for Line Integrals: The Forms Version). *Let C be an oriented curve $\gamma(t)$ ($a \le t \le b$) in $\mathbb{R}^n$ with an oriented boundary that designates b as the final point and a as the initial point. Let f be a function defined on a neighborhood of C then*

$$\int_C \mathbf{d}f = f(\gamma(b)) - f(\gamma(a)).$$

Here, the exterior derivative is the gradient. We are also requiring that $\mathbf{d}f$ be exact so the potential function f can be constructed. Hence, if constructed appropriately, $\mathbf{d}$ can be viewed as a conservative vector field.

3.5.3 Green's Theorem

We now express Green's theorem using work forms.

Theorem 15 (Green's Theorem: Work Forms Version). *Suppose that S is a bounded region in the plane whose boundary consists of some number of curves C_j. Give the union of these curves the boundary orientation (for simplicity, the counterclockwise diretion). Let $\mathbf{F}$ be a vector field defined on a neighborhood of S. Then*

$$\int_S \mathbf{d}W_{\mathbf{F}} = \sum_{1 \leq j \leq n} \int_{C_j} W_{\mathbf{F}}.$$

This still amounts to writing $P\,dx + Q\,dy$ in the right integrand and $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ in the left integrand. We still also have the fact that if the work form is an exact differential, the integral is zero.

3.5.4 Stokes' Theorem for Three Dimensions

A forms version of the traditional Stokes' theorem uses flux forms.

Theorem 16 (Traditional Stokes' Theorem: The Forms Version). *Let S be an oriented surface in $\mathbb{R}^3$ (usually, by the outward normal) bounded by a curve C given the boundary orientation (again, usually counterclockwise). Let φ be a 1-form field defined on a neighborhood of S. Then*

$$\int_S \mathbf{d}\varphi = \int_C \varphi.$$

Since φ is a 1-form, it represents a work form. The exterior derivative of such a form is a flux form. Flux integrals are surface integrals, and the function on the right is actually the curl of the vector field represented by φ.

3.5.5 *Divergence Theorem*

The divergence theorem, when written using forms, looks similar to the forms version of Green's theorem.

Theorem 17 (Divergence Theorem: The Forms Version). *Let X be a bounded domain in $\mathbb{R}^3$ with the standard orientation of space (this is given by the determinant of a change of basis matrix; usually this does not affect the integral). Let ∂X be a union of surfacees S_j each oriented by an outward normal. If φ is a 2-form field defined on a neighborhood of X, then*

$$\int_X \mathbf{d}\varphi = \sum_{1 \leq j \leq n} \int_{S_j} \varphi.$$

This theorem still states to integrate the surface's boundary using the divergence of a vector field in the integrand. This vector field comes from the flux form on the right side, and the divergence appears in the mass form on the left.

4

MAXWELL'S EQUATIONS

In electromagnetism, the behavior of electric fields, magnetic fields, and closed surfaces can be characterized by a set of four equations. These equations, developed by James Clerk Maxwell (for whom the equations are named for), describe both the flux of magnetic and electric fields and the imaginary rotation of these fields. Appropriately, the divergence and curl operators are used to describe the respective property.

There are two ways Maxwell's equations are traditionally written. One formulation uses partial differential equations that arise from the vector operators, while another expresses the same quantities as integral equations. In order to write these equations, we must first define three vector fields that exist in space-time.

The first field is **E**, the electric field. It is defined as the force a test charge experiences at a point divided by the charge itself. In symbols,

$$\mathbf{E} = \frac{\mathbf{F}}{q_0},$$

where **F** is the force the test charge experiences in Newtons and q_0 is the charge in coulombs (Serway, 688).

The second vector field is the magnetic field **B**. To define this field, it is useful to examine the force exerted by this field. Suppose a charge q moves with velocity **v** in a magnetic field **B**. Then the force in this magnetic field can be described by the Lorentz force law:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

The force then acts in a direction normal to the plane containing the velocity vector and the magentic field at a point. The magnitude of the magnetic field at a point can be expressed using the formula for the magnitude of the cross product; this gives $|\mathbf{B}| = \frac{|\mathbf{F}|}{q|\mathbf{v}|\sin\theta}$, where θ is the angle between the vectors **B** and **v**. Since the force is normal to the magnetic field and the velocity, the work done by **B** is therefore zero since no component of force is in the direction of motion.

Note that in the presence of an electric field, the Lorentz law can be written as $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ (Griffiths, 204).

The last field needed to generate Maxwell's equations is the current density **J**. This density is measured in statcolumbs per cubic centimeter in the centimeter-gram-second system. This vector field appears in the Maxwell-Ampere law of Maxwell's equations.

4.1 THE MAXWELL EQUATIONS - TRADITIONAL FORMS

There are four equations that characterize electricity and magnetism. There are two tradional ways that the Maxwell equations can be written.

4.1.1 *Partial Differential Equations*

It is common to see the Maxwell equations written as partial differential equations via vector operators. In this fashion, the equations are as follows, with c representing the speed of light and ρ the charge density:

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl} \mathbf{E}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \operatorname{curl} \mathbf{B} - \frac{4\pi \mathbf{J}}{c}$$
$$\nabla \cdot \mathbf{E} = 4\pi \rho$$

The first equation is Faraday's law; the second is Gauss's law in magnetic fields; the third, Ampere's law; the fourth, Gauss's law in electric fields. This representation is given in the Hubbard text.

4.1.2 *Integral Equations*

Some texts, such as Serway's *Physics for Scientists and Engineers* also give integral representations for Maxwell's equations. The integrals can be checked for validity using both Stokes' and Gauss' theorems in the appropriate contexts.

It is easy to obtain an integral representation for the second equation. Taking a triple integral on both sides and applying the divergence theorem backwards gives a surface integral of the magnetic field equal to zero.

For the fourth equation, the divergence theorem produces a surface integral of the electric field on the left side and a triple integral of the charge density on the right. But the integral of a density function gives the total amount of the measurement - in this case, charge. So,

$$\iint_{\sigma} \mathbf{E} \cdot d\mathbf{S} = 4\pi q,$$

or in SI units, $\frac{q}{\epsilon_0}$.

The curl equations are slightly more difficult to obtain an integral representation for. First, consider the integral

$$\oint_C \mathbf{E} \cdot d\mathbf{r}$$

taken over a closed curve C. Assuming the conditions for Stokes' theorem hold for $\mathbf{E}$ and C,

$$\oint_C \mathbf{E} \cdot d\mathbf{r} = \iint_\sigma \operatorname{curl} \mathbf{E} \cdot d\mathbf{S}$$

$$= \iint_\sigma -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$= -\frac{1}{c}\frac{d}{dt}\iint_\sigma \mathbf{B} \cdot d\mathbf{S}$$

where the last equality follows from differentiation under the integral sign. As a result, we have that the work done by the electric field along a closed curve C is proportional to the time derivative of the flux of $\mathbf{B}$ through the surface S bounded by C.

To obtain an integral form for the remaining equation, begin with the line integral of $\mathbf{B}$ over some closed curve C. Again, under appropriate conditions, the following holds:

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \iint_\sigma \operatorname{curl} \mathbf{B} \cdot d\mathbf{S}$$

$$= \iint_\sigma \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi \mathbf{J}}{c} \cdot d\mathbf{S}$$

$$= -\frac{1}{c^2}\frac{d}{dt}\iint_\sigma \mathbf{B} \cdot d\mathbf{S} + \frac{16\pi^2 I}{c^2}$$

where I represents the total current. Again, the derivative of flux appears in this integral representation.

4.2 DIFFERENTIAL FORMS AND MAXWELL'S EQUATIONS

We now come to a large application of differential forms. Maxwell's equations can be written concisely in two lines using three differential forms. Define the following forms:

1. The Faraday 2-form $\mathbb{F} = W_{\mathbf{E}} \wedge c\, dt + \Phi_{\mathbf{B}}$

2. The Maxwell 2-form $\mathbb{M} = W_{\mathbf{B}} \wedge c\, dt - \Phi_{\mathbf{E}}$

3. The current 3-form $\mathbb{J} = \frac{1}{c}\Phi_{\mathbf{J}} \wedge c\, dt - M_\rho$

We will use the exterior derivative and the differential equations version of Maxwell's equations to derivative the differential forms version.

4.2.1 *The Faraday Form*

Begin with the Faraday form $\mathbb{F}$ and take its exterior derivative.

$$
\begin{aligned}
\mathbf{d}\mathbb{F} &= \Phi_{\nabla\times\mathbf{E}} \wedge c\,dt + \mathbf{d}\Phi_{\mathbf{B}} \\
&= \Phi_{\nabla\times\mathbf{E}} \wedge c\,dt + M_{\nabla\cdot\mathbf{B}} + \Phi_{\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}} \wedge c\,dt \\
&= \left(\Phi_{\nabla\times\mathbf{E}+\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}}\right) \wedge c\,dt + M_{\operatorname{div}\mathbf{B}} \\
&= 0
\end{aligned}
$$

Each equality used either the derivatives of the basic forms or the partial differential equations from Maxwell. Thus, for two equations, we have the compact formulation $\mathbf{d}\mathbb{F} = 0$.

4.2.2 *The Maxwell Form*

We use the same process as above to find the exterior derivative of the Maxwell form.

$$
\begin{aligned}
\mathbf{d}\mathbb{M} &= \Phi_{\nabla\times\mathbf{B}} \wedge c\,dt - M_{\nabla\cdot\mathbf{E}} - \Phi_{\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}} \wedge c\,dt \\
&= 4\pi\left(\Phi_{\frac{\mathbf{J}}{c}} \wedge c\,dt - M_{\rho}\right) \\
&= 4\pi\mathbb{J}
\end{aligned}
$$

The last line relied on converting to the current form $\mathbb{J}$ to condense the equation. Thus the remaining two of Maxwell's equations can be written as $\mathbf{d}\mathbb{M} = 4\pi\mathbb{J}$.

Differential forms allow for many concepts in electromagnetism to be simplified by separating spatial variables and the time variable. The use of differential forms in physics can be extended to quantum mechanics, as seen in Garrity.

REFERENCES

Fleisch, D. (2008). A Student's Guide to Maxwell's Equations. Cambridge University Press.

Garrity, T. A. (2015). Electricity and Magnetism for Mathematicians. Cambridge University Press.

Griffiths, D. J. (1999). Introduction to Electrodynamics (3rd ed.). Prentice Hall.

Hubbard, J. H., & Hubbard, B. B. (2009). Vector Calculus, Linear Algebra, and Differential Forms (4th ed.). Matrix Editions.

Serway, R. A., & John W. Jewett, J. (2008). Physics for Scientists and Engineers with Modern Physics (7th ed.). Brooks/Cole.

Stewart, J. (2012). Calculus: Early Transcendentals (7th ed.). Brooks/Cole.

Strauss, W. A. (2008). Partial Differential Equations: An Introduction (2nd ed.).

Zill, D. G. (2009). A First Course in Differential Equations with Modeling Applications (9th ed.). Brooks/Cole.