

SERIES CONVERGENCE/DIVERGENCE FLOW CHART

TEST FOR DIVERGENCE

Does $\lim_{n \rightarrow \infty} a_n = 0$?

NO

$\sum a_n$ Diverges

YES

p-SERIES

Does $a_n = 1/n^p, n \geq 1$?

YES

Is $p > 1$?

YES

$\sum a_n$ Converges

NO

$\sum a_n$ Diverges

NO

GEOMETRIC SERIES

Does $a_n = ar^{n-1}, n \geq 1$?

YES

Is $|r| < 1$?

YES

$\sum_{n=1}^{\infty} a_n = \frac{a}{1-r}$

NO

$\sum a_n$ Diverges

NO

ALTERNATING SERIES

Does $a_n = (-1)^n b_n$ or $a_n = (-1)^{n-1} b_n, b_n \geq 0$?

YES

Is $b_{n+1} \leq b_n$ & $\lim_{n \rightarrow \infty} b_n = 0$?

YES

$\sum a_n$ Converges

NO

TELESCOPING SERIES

Do subsequent terms cancel out previous terms in the sum? May have to use partial fractions, properties of logarithms, etc. to put into appropriate form.

YES

Does $\lim_{n \rightarrow \infty} s_n = s$ finite?

YES

$\sum a_n = s$

NO

$\sum a_n$ Diverges

NO

TAYLOR SERIES

Does $a_n = \frac{f^{(n)}(a)}{n!} (x-a)^n$?

YES

Is x in interval of convergence?

YES

$\sum_{n=0}^{\infty} a_n = f(x)$

NO

$\sum a_n$ Diverges

NO

Try one or more of the following tests:

COMPARISON TEST

Pick $\{b_n\}$. Does $\sum b_n$ converge?

YES

Is $0 \leq a_n \leq b_n$?

YES

$\sum a_n$ Converges

NO

Is $0 \leq b_n \leq a_n$?

YES

$\sum a_n$ Diverges

LIMIT COMPARISON TEST

Pick $\{b_n\}$. Does $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c > 0$ finite & $a_n, b_n > 0$?

YES

Does $\sum_{n=1}^{\infty} b_n$ converge?

YES

$\sum a_n$ Converges

NO

$\sum a_n$ Diverges

INTEGRAL TEST

Does $a_n = f(n), f(x)$ is continuous, positive & decreasing on $[a, \infty)$?

YES

Does $\int_a^{\infty} f(x) dx$ converge?

YES

$\sum_{n=a}^{\infty} a_n$ Converges

NO

$\sum a_n$ Diverges

RATIO TEST

Is $\lim_{n \rightarrow \infty} |a_{n+1}/a_n| \neq 1$?

YES

Is $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$?

YES

$\sum a_n$ Abs. Conv.

NO

$\sum a_n$ Diverges

ROOT TEST

Is $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} \neq 1$?

YES

Is $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$?

YES

$\sum a_n$ Abs. Conv.

NO

$\sum a_n$ Diverges

Summary of Convergence Tests for Series

Test	Series	Convergence or Divergence	Comments
n^{th} term test (or the zero test)	$\sum a_n$	Diverges if $\lim_{n \rightarrow \infty} a_n \neq 0$	Inconclusive if $\lim_{n \rightarrow \infty} a_n = 0$.
Geometric series	$\sum_{n=0}^{\infty} ax^n$ (or $\sum_{n=1}^{\infty} ax^{n-1}$)	Converges to $\frac{a}{1-x}$ only if $ x < 1$ Diverges if $ x \geq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to ax^n .
p -series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	Converges if $p > 1$ Diverges if $p \leq 1$	Useful for comparison tests if the n^{th} term a_n of a series is similar to $\frac{1}{n^p}$.
Integral	$\sum_{n=c}^{\infty} a_n$ ($c \geq 0$) $a_n = f(n)$ for all n	Converges if $\int_c^{\infty} f(x) dx$ converges Diverges if $\int_c^{\infty} f(x) dx$ diverges	The function f obtained from $a_n = f(n)$ must be continuous, positive, decreasing and readily integrable for $x \geq c$.
Comparison	$\sum a_n$ and $\sum b_n$ with $0 \leq a_n \leq b_n$ for all n	$\sum b_n$ converges $\implies \sum a_n$ converges $\sum a_n$ diverges $\implies \sum b_n$ diverges	The comparison series $\sum b_n$ is often a geometric series or a p -series.
Limit Comparison*	$\sum a_n$ and $\sum b_n$ with $a_n, b_n > 0$ for all n and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$	$\sum b_n$ converges $\implies \sum a_n$ converges $\sum b_n$ diverges $\implies \sum a_n$ diverges	The comparison series $\sum b_n$ is often a geometric series or a p -series. To find b_n consider only the terms of a_n that have the greatest effect on the magnitude.
Ratio	$\sum a_n$ with $\lim_{n \rightarrow \infty} \frac{ a_{n+1} }{ a_n } = L$	Converges (absolutely) if $L < 1$ Diverges if $L > 1$ or if L is infinite	Inconclusive if $L = 1$. Useful if a_n involves factorials or n^{th} powers.
Root*	$\sum a_n$ with $\lim_{n \rightarrow \infty} \sqrt[n]{ a_n } = L$	Converges (absolutely) if $L < 1$ Diverges if $L > 1$ or if L is infinite	Test is inconclusive if $L = 1$. Useful if a_n involves n^{th} powers.
Absolute Value $\sum a_n $	$\sum a_n$	$\sum a_n $ converges $\implies \sum a_n$ converges	Useful for series containing both positive and negative terms.
Alternating series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ ($a_n > 0$)	Converges if $0 < a_{n+1} < a_n$ for all n and $\lim_{n \rightarrow \infty} a_n = 0$	Applicable only to series with alternating terms.

*The Root and Limit Comparison tests are not included in the current textbook used in Calculus classes at Bates College.